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# THE DYNAMIC PLASTIC RESPONSE OF A TUBE TO AN IMPULSIVE RING LOAD OF ARBITRARY PULSE SHAPE

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Carl K. Youngdahl

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Reactor Engineering Division

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# THE DYNAMIC PLASTIC RESPONSE OF A TUBE TO AN IMPULSIVE RING LOAD OF ARBITRARY PULSE SHAPE

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## ABSTRACT

The problem of the deformation of a long, circular, cylindrical shell made of a rigid, perfectly plastic material and subjected to an impulsive ring load is solved for arbitrary pulse shapes. The principal result of the investigation is that the dynamic effect of the pulse shape is almost completely characterized by its associated impulse and effective load (defined as the impulse divided by twice the mean time of the pulse). This is to say that details of the pulse shape and, in particular, its peak value are comparatively unimportant in determining the history of the plastic deformation and the final shape of the shell. The implications for experimental work are encouraging, since the impulse and effective load are easily computed from a pressure-time plot and are not strongly influenced by experimental inaccuracies.

## INTRODUCTION

In nuclear-reactor safety analyses it is frequently important to estimate the plastic deformation of pipes and tubes in the reactor produced by small chemical, mechanical, or nuclear explosions. For example, proof tests of fuel-element prototypes are carried out in experimental loops in special testing reactors in which the fuel elements to be tested are deliberately caused to fail. It is necessary to predict the extent to which the resultant localized release of energy will distort the test loop if damage to the reactor itself is to be avoided. Another example of reactor analysis pertaining to plastic distortion of tubes is the case of a coolant-flow blockage producing a localized fuel-element failure. The impulsive pressure may damage the fuel-assembly can, and this may lead to a more serious accident or, at the least, may jam the can into its neighbors, creating repair and removal problems.

The nature of the energy-release phenomena in these applications is usually only crudely known, and out-of-pile simulations mockup only part of the complex nuclear and structural configurations of a reactor. Consequently, it would be advantageous to know what properties of the time-shape of an impulsive load are important in determining the final plastic deformation of a tube; in particular, for experimental applications, one should know what scaling and equivalence laws may be used and how detailed and accurate pressure-time measurements must be.

For a free-free beam made of a rigid-plastic material, Symonds[1] compared the final deformations produced by triangular, sinusoidal, and rectangular pulse shapes and postulated that pulses with the same total impulse and peak value had approximately the same effect. Hodge,[2] however, found for a reinforced circular cylindrical shell that this equivalence was reasonably adequate if the yield load was greatly exceeded, but that the final deformations were strongly dependent on the pulse shape for "medium" values of the load, which are of the same order of magnitude as the collapse load. In particular, he showed the deformation produced by an exponentially decaying pulse, which is a reasonable approximation to an actual explosive load, differed significantly from that produced by a rectangular pulse with the same impulse and peak value.

This report shows that, for a long, circular cylinder made of a rigid plastic material and acted on by an impulsive ring load, the final plastic deformation is almost completely determined by the impulse and <u>effective load</u> associated with the pulse. The <u>effective load</u> is defined as the impulse divided by twice the time-mean of the pulse (the time-mean being the length of the interval between the time at which yielding occurs and the centroid of the pulse shape). This correlation has been found to be valid for exponentially decaying, triangular, rectangular, ramp, multisaw-toothed, and other widely varying pulse shapes. Both parameters are easily determined from experimental pressure-time graphs, in contrast to the measurement of the peak load, which is strongly influenced by transducer inaccuracies.

The dynamic plasticity problem treated here has been solved by Eason and Shield[3] for the special cases of a rectangular pulse, for which they obtained a closed-form solution, and a triangular pulse, which they treated numerically. Their work is extended here to the consideration of arbitrary pulse shapes. An elaborate computer program was devised to numerically solve the resulting sets of coupled nonlinear differential equations for any pulse-shape input. However, if the error involved in applying the two-parameter correlation is acceptable (and it is apparently less than the errors made in idealizing the material behavior), a semigraphical procedure may be used in place of the computer analysis.

The basic equations for the plastic deformation of a long tube acted on by an impulsive ring load are derived in Ref. 3 in considerable detail. A condensed version of this derivation is given in the next section to make this report reasonably self-contained.

## GENERAL EQUATIONS

Consider a long, circular, cylindrical tube, made of a rigid, perfectly plastic material and loaded by an internal pressure $P(Z,T)$, where $Z$ and $T$ are axial coordinate and time. The usual shell-theory assumptions are employed such that stress distributions across the shell thickness are replaced by their resultants per unit circumferential length as in Fig. 1. The equations of motion of the shell can be expressed in terms of the axial bending moment $M$, the radial shear force $Q$, the circumferential force $N$, the radial velocity $V$, and the radial displacement $U$. Define dimensionless quantities by

$$\left.\begin{aligned} z &= \frac{Z}{(RH)^{1/2}}, \quad t = \frac{T}{T_0}, \quad u = \frac{UR\rho}{\sigma_y H T_0^2}, \quad v = \frac{VR\rho}{\sigma_y H T_0}, \quad m = \frac{4M}{\sigma_y H^2}, \\ n &= \frac{N}{\sigma_y H}, \quad q = \frac{QR^{1/2}}{\sigma_y H^{3/2}}, \quad \text{and} \quad p = \frac{PR}{\sigma_y H}, \end{aligned}\right\} \tag{1}$$

where $R$ and $H$ are the radius and thickness, of the shell as shown in Fig. 2, $\sigma_y$ and $\rho$ are its yield stress and surface density, and $T_0$ is a time interval characteristic of the problem under discussion.† The equations of motion are then

$$\frac{\partial q}{\partial z} = p - n - \frac{\partial v}{\partial t}, \quad \frac{\partial m}{\partial z} = 4q, \quad \text{and} \quad \frac{\partial u}{\partial t} = v, \tag{2}$$

with all dependent variables being functions of $z$ and $t$.

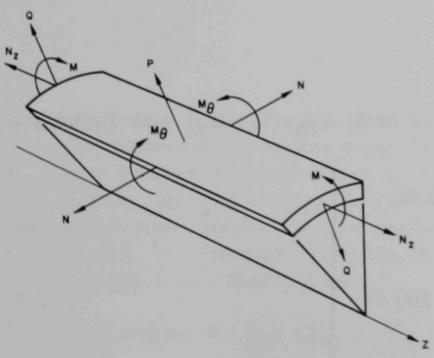


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Fig. 1. Stress Resultants on Element of Shell

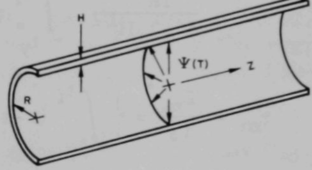


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Fig. 2. Circular Cylindrical Shell Subjected to an Impulsive Ring Load

†Any value of $T_0$ may be chosen, since the results are plotted so as to eliminate this quantity.

To obtain a concentrated ring load, let the region of application of the internal pressure shrink to the plane $Z = 0$, maintaining the value of the resultant force as a function of time. Then

$$P(Z,T) = \Psi(T)\delta(Z), \tag{3}$$

where $\delta$ is the Dirac delta function. The dimensionless ring-load magnitude $\psi(t)$ is defined by

$$\psi = \frac{\Psi R^{1/2}}{2\sigma_y H^{3/2}}. \tag{4}$$

Some quantities of interest associated with $\Psi(T)$ are the impulse I per unit circumferential length and the time mean $T_m$ of the pulse, defined by

$$I = \int_{T_y}^{T_f} \Psi(T)\, dT,$$

and

$$T_m = \frac{1}{I}\int_{T_y}^{T_f} (T - T_y)\,\Psi(T)\, dT, \tag{5}$$

where $T_y$ and $T_f$ are the times at which plastic deformation begins and ends. It is proposed that the parameter that, along with I, best characterizes the effect of the pulse, is the associated effective load $\Psi_e$ defined by

$$\Psi_e = \frac{I}{2T_m}. \tag{6}$$

Let the dimensionless counterparts of I, $T_m$, $\Psi_e$, $T_y$, and $T_f$ be given by

$$\left.\begin{aligned} i &= \frac{IR^{1/2}}{2\sigma_y T_0 H^{3/2}} = \int_{t_y}^{t_f} \psi(t)\, dt, \\ t_m &= \frac{T_m}{T_0} = \frac{1}{i}\int_{t_y}^{t_f} (t - t_y)\,\psi(t)\, dt \\ \psi_e &= \frac{IR^{1/2}}{4T_m \sigma_y H^{3/2}} = \frac{i}{2t_m}, \\ t_y &= \frac{T_y}{T_0}, \text{ and } t_f = \frac{T_f}{T_0}. \end{aligned}\right\} \tag{7}$$

Assume the yield condition in m,n space to be given by the limited interaction curve of Fig. 3. (The relation of this ideal square curve to the true yield condition, is discussed by Drucker[4] and Hodge.[5]) The relevant points of the yield curve for this problem are on the side AB, including the end points, such that $n = +1$ throughout the plastic region. Since the strain-rate vector, which has components proportional to $\partial^2 v/\partial z^2$ and v, must be normal to the yield curve, it follows that the radial velocity v must be positive and that

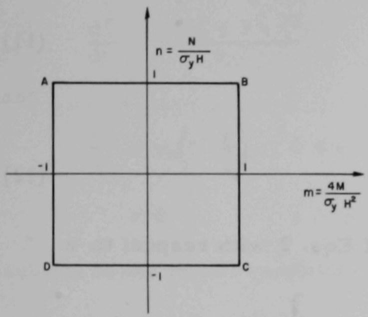


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Fig. 3. Limited-interaction Yield Condition

$$\frac{\partial^2 v}{\partial z^2} = 0 \tag{8}$$

at shell sections that correspond to interior points of AB. Hinge circles occur at sections of the shell for which $m = \pm 1$; they correspond to points A and B where the strain-rate vector can have any direction between the limiting normals, implying the possibility of a discontinuity in the velocity slope $\partial v/\partial z$. If a hinge circle spreads over a finite axial region of the shell, a hinge band is formed in which m has the corresponding extremum value and $\partial^2 v/\partial z^2$ may have any value compatible with the yield condition.

The static collapse load for a long shell acted on by a circumferential ring force is found in Ref. 4 to be

$$\Psi_s = \frac{2\sigma_y H^{3/2}}{R^{1/2}}. \tag{9}$$

For a dynamically applied load whose maximum value exceeds the collapse load, plastic deformation begins when (by Refs. 4 and 9) $\psi$ first exceeds unity. Hinge circles initially occur at $z = 0$ and $z = \pm 1$. During the subsequent motion, the outer hinge circles occupy the positions $z = \pm\zeta(t)$ and, for some load-time functions, may expand into hinge bands in the regions $\zeta(t) \le |z| \le \eta(t)$. We consider first the deformation history up until the time when hinge bands first appear.†

In the region $0 < z < \zeta(t)$, $t \ge t_y$, we have (from Eq. 8 and Fig. 3) that

$$v(z,t) = v_0(t)\left[1 - \frac{z}{\zeta(t)}\right], \text{ and } n(z,t) = 1, \tag{10}$$

†As shown in the section on Solutions, a hinge band cannot appear instantaneously at $t = t_y$ even if an instantaneous jump in the load occurs there.

where $v_0(t)$ is the radial velocity at $z = 0$. The first of Eqs. 2 can then be written

$$\frac{\partial q}{\partial z} = zC_1(t) - C_2(t), \tag{11}$$

with $C_1$ and $C_2$ defined by

$$C_1 = \frac{d}{dt}\left(\frac{v_0}{\zeta}\right), \text{ and } C_2 = 1 + \frac{dv_0}{dt}. \tag{12}$$

The integrations of Eq. 11 and the second of Eqs. 2 with respect to z result in

$$\left.\begin{aligned} q &= \tfrac{1}{2}z^2C_1(t) - zC_2(t) + C_3(t), \\ &\text{and} \\ m &= \tfrac{2}{3}z^3C_1(t) - 2z^2C_2(t) + 4zC_3(t) + C_4(t), \\ &\text{for } 0 \le z \le \zeta(t), \quad t \ge t_y. \end{aligned}\right\} \tag{13}$$

The boundary conditions on m are (using Fig. 3)

$$m(0,t) = -1, \quad m(\zeta,t) = +1, \tag{14}$$

while $q = 0$ at $z = \zeta$ where m has its maximum and q has a discontinuity of $2\psi$ at $z = 0$ because of the concentrated load there. Applying these boundary conditions to Eqs. 13 gives four relations for $C_1$, $C_2$, $C_3$, and $C_4$, whose solution is

$$\left.\begin{aligned} C_1 &= \frac{6}{\zeta^3}(\psi\zeta - 1), \\ C_2 &= \frac{1}{\zeta^2}(4\psi\zeta - 3), \\ C_3 &= \psi, \\ &\text{and} \\ C_4 &= -1. \end{aligned}\right\} \tag{15}$$

Using Eqs. 10, 12, and 15 and the third of Eqs. 2, we arrive at the set of coupled nonlinear differential equations

$$\left.\begin{aligned}
\frac{dv_0}{dt} &= \frac{4\psi\zeta - 3 - \zeta^2}{\zeta^2}, \\
\frac{d\zeta}{dt} &= \frac{-2\psi\zeta + 3 - \zeta^2}{v_0\zeta}, \\
\text{and} \qquad & \\
\frac{\partial u}{\partial t} &= v_0\left(1 - \frac{z}{\zeta}\right), \qquad 0 \le z \le \zeta(t), \\
&= 0, \qquad z > \zeta(t),
\end{aligned}\right\} \tag{16}$$

subject to the initial conditions

$$v_0(t_y) = 0, \quad \zeta(t_y) = 1, \quad \text{and} \quad u(z,t_y) = 0. \tag{17}$$

Since m is a cubic in z, satisfies the boundary conditions (Eqs. 14), and has a horizontal tangent at $z = \zeta$, necessary and sufficient conditions that $|m| < 1$ in the interval $0 < z < \zeta$ are that

$$\frac{\partial m}{\partial z} \ge 0 \tag{18}$$

at $z = 0$, and

$$\frac{\partial^2 m}{\partial z^2} \le 0 \tag{19}$$

at $z = \zeta$. The slope of m at the origin is $\psi(t)$. Since $\psi(t)$ is assumed to be nonnegative, the first of these conditions is always satisfied. This implies that a hinge band† cannot form in the vicinity of $z = 0$. By Eqs. 13 and 15, the Inequality 19 is equivalent to

$$\psi\zeta \le 3/2. \tag{20}$$

For some load histories, the deformation behavior is such that at some time $t = t_1$, say, while $\psi$ is increasing,

$$\psi\zeta = 3/2. \tag{21}$$

At this instant, the hinge circle at $\zeta(t)$ begins to broaden into a hinge band in the region $\zeta(t) \le z \le \eta(t)$. The hinge band continues to broaden until $t = t_M$ when $\psi$ reaches a local maximum $\psi_M$. As $\psi$ decreases, the hinge band contracts, until $\zeta$ and $\eta$ coincide at some time $t_2$ when Eqs. 16 are again applicable. If $\psi$ has a number of peaks, the hinge band may not

† It is shown in Ref. 3 that a hinge band can form at the origin if the load is axially distributed, rather than concentrated.

completely disappear between adjacent maxima, but begin to expand as $\psi$ passes through the intervening minimum.

During the existence of the hinge band, the shell radial velocity in the region $0 \le z \le \zeta(t)$ is still linear in $z$, but is given by

$$v(z,t) = v_0(t) - [v_0(t) - v_\zeta(t)] \frac{z}{\zeta(t)}, \tag{22}$$

where $v_\zeta(t)$ is the radial velocity at $z = \zeta$. The solution to the first two equations of Eqs. 2 is still given by Eqs. 13 and 15, where now $C_1$ and $C_2$ are defined as

$$C_1 = \frac{d}{dt} \frac{v_0 - v_\zeta}{\zeta}, \text{ and } C_2 = 1 + \frac{dv_0}{dt}. \tag{23}$$

In the region $\zeta(t) \le z \le \eta(t)$,

$$m = 1, \ q = 0, \text{ and } n = +1, \tag{24}$$

so that the solution to the first two equations of Eqs. 2 is

$$v(z,t) = -t + \Omega(z), \tag{25}$$

where $\Omega(z)$ is an arbitrary function determined by the deformation. Since the radial velocity at the outer end of the hinge band vanishes, we have that

$$\Omega(\eta) = t, \tag{26}$$

while by definition of $v_\zeta$,

$$v_\zeta = -t + \Omega(\zeta). \tag{27}$$

It is shown in Ref. 3 that in the interval $t_1 \le t \le t_M$ the inner edge of the hinge band $\zeta$ is related to $\psi$ through Eq. 21. The pair of nonlinear differential equations obtained from Eqs. 23 and 15 can then be solved for $v_0(t)$ and $v_\zeta(t)$. Moreover, since $\psi$ is increasing in this time interval, $\zeta$ must be decreasing; that is, the inner end of the hinge band moves toward the origin. As it does so, the function $\Omega$ is generated through Eq. 27. The outer end of the hinge band $\eta$ is then determined by Eq. 26. In the interval $t_M \le t \le t_2$, $\zeta$ and $\psi$ are no longer related by Eq. 21, but $\zeta$ now increases with time so that it sweeps back through positions $z$ for which $\Omega$ is known. Consequently Eqs. 23, 15, and 27 give three equations for the determination of $\zeta$, $v_0$, and $v_\zeta$, while $\eta$ is still found from Eq. 26.† Since $\zeta$ is

† It is easily shown that $\Omega$ is a decreasing function of $z$ so that, by Eq. 26, $\eta$ must be a decreasing function of time. Consequently, $\eta$ passes through positions $z$ previously occupied by $\zeta$ for which $\Omega(z)$ has been determined.

increasing and $\eta$ is decreasing, they eventually coincide and the hinge band reverts to a hinge circle.

A function $\Omega$ must be determined for each hinge band if more than one occurs for a $\psi$ function with several peaks. If between adjacent peaks the hinge band begins expanding again, rather than degenerating to zero length, the function $\Omega(z)$ must be redefined at locations that are passed through by $\zeta$ a second time.

A closed-form solution for $v_0$, $v_\zeta$, $\zeta$, $\eta$, and $\Omega$ during hinge formation and motion for arbitrary† $\psi(t)$ is derived in Appendix A. However, in obtaining results for the shell deformation, we decided that solving the differential equations numerically was probably no more difficult than numerically evaluating the transcendental equations of the closed-form solution and could be adapted more readily to the solution of multiple-peak problems.

## SOLUTIONS

A numerical solution of the coupled set of nonlinear differential equations (Eqs. 16) and the set, for hinge-band motion, given by Eqs. 15, 22, 23, 25, 26, and 27 was programmed for the CDC-3600 computer for an arbitrary load-time function $\psi(t)$. The program and auxiliary subroutines are given in Appendix D. The inputs to the program are the function $\psi$ and the time intervals and axial positions at which results are desired; the outputs are radial velocities and displacements and hinge-band locations. The Bulirsch-Stoer method of extrapolation by rational functions was used,[6] an initial power series being employed to facilitate the start of the solution. (See Appendix B.)

A closed-form solution may be obtained for a rectangular pulse shape. Let $\psi$ be given by

$$\left.\begin{aligned} \psi &= \psi_M, \quad 0 \le t \le \tau; \\ &= 0, \quad t > \tau. \end{aligned}\right\} \tag{28}$$

At $t = 0$, the hinge-circle location jumps instantaneously from its yield location at $z = 1$ to the position $z_0$. During the interval $0 < t \le \tau$, the hinge-circle location remains fixed while $v_0$ is a linear function of time. From Eqs. 16, we have, for $0 < t \le \tau$,

$$\zeta(t) = z_0, \text{ and } v_0(t) = Kt, \tag{29}$$

† The particular case of this solution for $\psi$ a triangular peak is presented in Ref. 3.

with

$$\left.\begin{aligned} z_0 &= \left(\psi_M^2 + 3\right)^{1/2} - \psi_M, \\ \text{and} \\ K &= \frac{3(1 - z_0^2)}{z_0^2}. \end{aligned}\right\} \tag{30}$$

The solution to Eqs. 16 for any $\psi(t)$ that vanishes after some time $\tau$ is given by

$$\zeta = A\frac{t}{\tau} + B, \quad v_0 = \frac{(3 - \zeta^2)\,\tau}{A\zeta}, \qquad t \geq \tau, \tag{31}$$

where A and B are found from the values of $\zeta$ and $v_0$ at $\tau$. By Eqs. 29 and 30, for the rectangular pulse,†

$$A = \frac{(3 - z_0^2)\, z_0}{3(1 - z_0^2)}, \text{ and } B = z_0 - A. \tag{32}$$

From Eqs. 16 and 29-32, the radial displacement is found to be, for $0 \leq t \leq \tau$,

$$\left.\begin{aligned} u(z,t) &= \frac{1}{2}K\left(1 - \frac{z}{z_0}\right)t^2, \qquad 0 \leq z \leq z_0, \\ &= 0, \qquad z > z_0, \end{aligned}\right\} \tag{33}$$

while, for $t > \tau$,

$$\left.\begin{aligned} \frac{u(z,t)}{\tau^2} &= \frac{1}{2}K\left(1 - \frac{z}{z_0}\right) + \frac{1}{A^2}\left\{\frac{1}{2}[z_0 - \zeta(t)][z_0 + \zeta(t) - 2z] \right. \\ &\quad \left. + 3 \log\frac{\zeta(t)}{z_0} + 3z\left[\frac{1}{\zeta(t)} - \frac{1}{z_0}\right]\right\}, \qquad 0 \leq z \leq z_0; \\ &= \frac{1}{A^2}\left\{-\frac{1}{2}[\zeta(t) - z]^2 + 3 \log\frac{\zeta(t)}{z} + 3\left[\frac{z}{\zeta(t)} - 1\right]\right\}, \\ &\quad z_0 < z \leq \zeta; \\ &= 0, \qquad z > \zeta, \end{aligned}\right\} \tag{34}$$

†The results (Eqs. 29-32) are presented in Ref. 3, in somewhat different form.

with $\zeta(t)$ and A given by Eqs. 31 and 32. The plastic deformation ends when $v_0$ vanishes, which, by Eqs. 31, occurs at a time $t_f$ when

$$\zeta(t_f) = \sqrt{3}. \tag{35}$$

Letting $u_{0f}$ be the radial displacement at $z = 0$ and $t = t_f$, we have, from Eqs. 29-32, 34, 35, and 7, that

$$\left.\begin{aligned} \frac{t_f}{i} &= \frac{\sqrt{3} - B}{A\psi_M}, \\ &\text{and} \\ \frac{u_{0f}}{i^2} &= \frac{1}{\psi_M^2 A^2}\left(z_0 A + 3 \log \frac{\sqrt{3}}{z_0}\right). \end{aligned}\right\} \tag{36}$$

Note that, for the rectangular pulse,

$$\psi_e = \psi_M. \tag{37}$$

It is easily shown that

$$\psi_M z_0 < 3/2, \tag{38}$$

so that no hinge band is produced by a rectangular pulse. The same is also true for any pulse shape that increases instantaneously from zero to its maximum value $\psi_M$ and then decreases monotonically from that point on. The initial instantaneous value of $\zeta$ will be $z_0$, given in Eqs. 30, for which Eq. 38 holds, and a hinge band can only be initiated when $\psi$ is increasing. However, if the pulse shape is such that the shell is already in motion when a positive jump in $\psi$ occurs, a hinge band may appear instantaneously. Consider a time $t_1$ at which $\zeta$ and $v_0$ have the values $\zeta_1$ and $v_{01}$ and at which $\psi$ jumps from $\psi_1$ to $\psi_M$, where $\psi_M \zeta_1$ exceeds $3/2$. Let primes denote quantities just after the jump. Then,

$$\left.\begin{aligned} &\eta_1' = \zeta_1, \;\; \zeta_1' = \frac{3}{2\psi_M}, \;\; v_{01}' = v_{01}, \;\; v_{\zeta_1}' = v_{01}\left(1 - \frac{\zeta_1'}{\zeta_1}\right), \\ &\Omega(\zeta_1) = t_1, \;\text{ and }\; \Omega(\zeta_1') = t_1 + v_{\zeta_1}', \end{aligned}\right\} \tag{39}$$

and $\Omega$ is a linear function of $z$ in the range $\zeta_1' \leq z \leq \zeta_1$. In Appendix C, these results are obtained by considering the series solution to the differential equations at some time $t_1$ and finding the appropriate limits as $\Delta t$ vanishes with $\Delta\psi$ remaining fixed.

## RESULTS AND CONCLUSIONS

Figures 4-6 show the results of some of the parameter studies. The solution for the rectangular pulse shape was discussed in the previous section; the linear-decay, triangular, and exponential-decay pulse shapes are given by

Linear Decay

$$\left.\begin{aligned} \psi &= \psi_M\left(1-\frac{t}{\tau}\right), && 0 \le t \le \tau; \\ &= 0, && t > \tau. \end{aligned}\right\} \tag{40}$$

Triangular

$$\left.\begin{aligned} \psi &= 2\psi_M \frac{t}{\tau}, && 0 \le t \le \tfrac{1}{2}\tau; \\ &= 2\psi_M\left(1-\frac{t}{\tau}\right), && \tfrac{1}{2}\tau < t \le \tau; \\ &= 0, && t > \tau. \end{aligned}\right\} \tag{41}$$

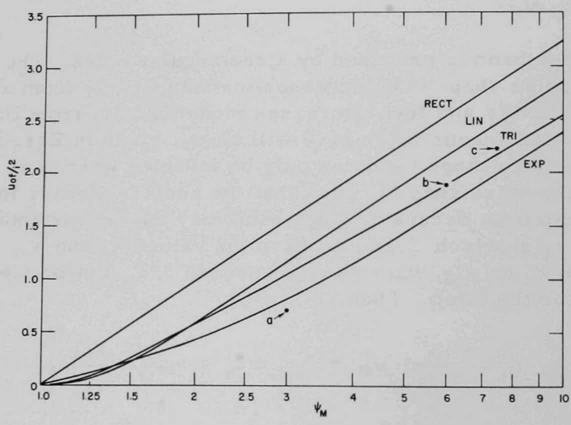


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Fig. 4. Final Deformation as a Function of Peak Load for Rectangular, Triangular, Linear-decay, and Exponential-decay Pulses. Points a, b, and c are the results of problems a, b, and c.

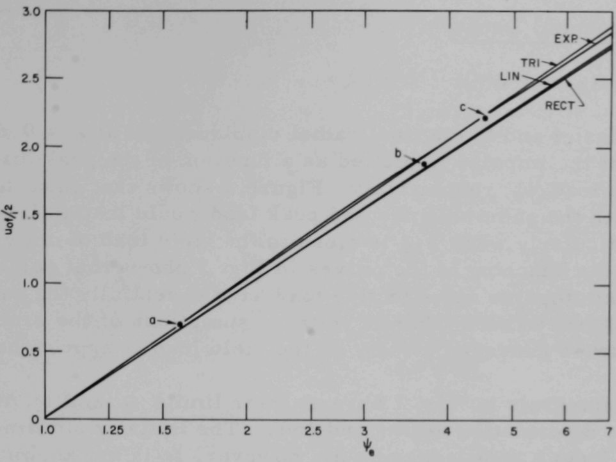


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Fig. 5. Final Deformation as a Function of Effective Load for the Same Pulse Shapes as Fig. 4

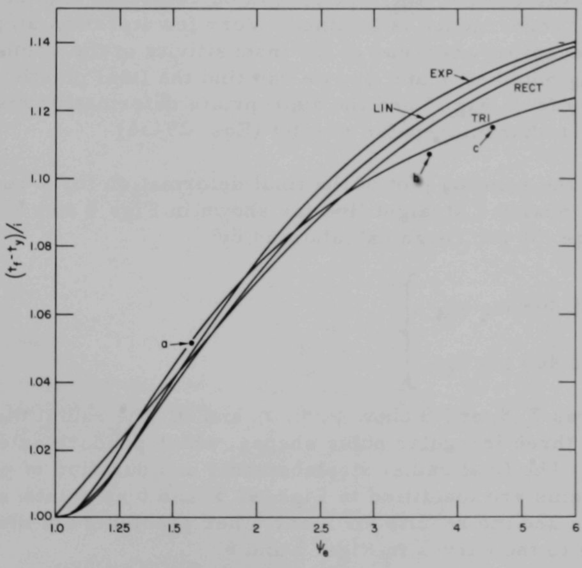


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Fig. 6. Duration of Plastic Flow as a Function of Effective Load for the Same Pulse Shapes as Fig. 4

Exponential Decay

$$\psi = \psi_M \exp(-t/\tau), \qquad 0 \le t < \infty. \tag{42}$$

In Figs. 4 and 5, the final radial displacement at $z = 0$ divided by the square of the impulse is plotted as a function of the peak load $\psi_M$ and the effective load $\psi_e$, respectively. Figure 4 shows that equating the effects of pulses with the same impulse and peak load would lead to large relative errors, particularly when $\psi_M$ is close to the yield load of unity. On the other hand, the bunching of the curves in Fig. 5 shows that pulse shapes with the same impulse and effective load have essentially the same effect, to well within the errors inherent in the assumptions of the problem. Moreover, the curves converge closely as the yield load is approached.

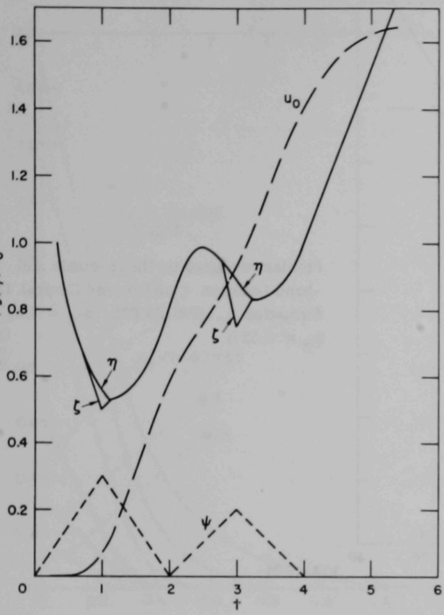
The integrals in Eqs. 7 have as their limits $t_y$ and $t_f$, the times when plastic deformation begins and ends. The initial yield time is easily determined from a knowledge of $\psi(t)$; however, $t_f$ is not known a priori since the motion may cease before the end of the pulse.† Figure 6 indicates that $(t_f - t_y)/i$ is a weak function of $\psi_e$. Consequently, given an arbitrary pulse shape, $t_f$ can be estimated and $i$, $t_m$, and $\psi_e$ computed using Eqs. 7. For these values of $i$ and $\psi_e$, a new value of $t_f$ can be found from Fig. 6. Then revised values of $i$, $t_m$, and $\psi_e$ can be computed and the procedure repeated until convergence is attained. Very few iteration steps have been found to be necessary, because of the insensitivity of the value of $(t_f - t_y)/i$ to $\psi_e$. Having obtained $i$ and $\psi_e$, we can find the final plastic deformation approximately from Fig. 5 and the appropriate deformation history from the closed-form, rectangular-pulse results (Eqs. 29-36).

Since the semilog plot of the final deformation for a rectangular pulse is very nearly a straight line (as shown in Figs 4 and 5), we can approximate Eqs. 36 for rough calculations by

$$\left.\begin{aligned} \frac{u_{0f}}{i^2} &\approx 1.405 \log \psi_M \\ &\approx 1.405 \log \psi_e . \end{aligned}\right\} \tag{43}$$

Figures 7, 8, and 9 show $\psi$, $\zeta$, $\eta$, and $u_0$ (the radial displacement at $z = 0$) for three irregular pulse shapes, which produce extensive hinge-band motion. The final radial displacements and duration of plastic flow for the problems are identified in Figs. 4, 5, and 6 as points a, b, and c. These results and the results for many other problems not discussed here cluster close to the curves in Figs. 5 and 6.

† In the results included here, this occurs for all the exponential-decay pulses, of course, and for the linear-decay and triangular cases with $\psi_e$ close to unity.




113-1886

Fig. 7

Problem a: Load $\psi$, Hinge-circle and -band Location $\zeta$ and $\eta$, and Central Deformation $u_o$ ($i = 4.833$, $\psi_M = 3$, $\psi_e = 1.589$)

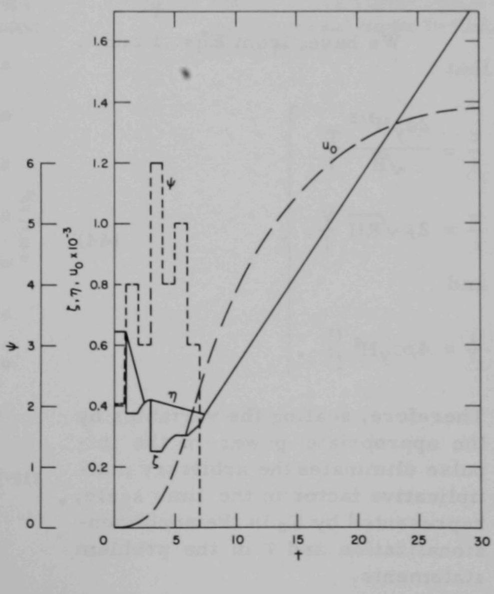
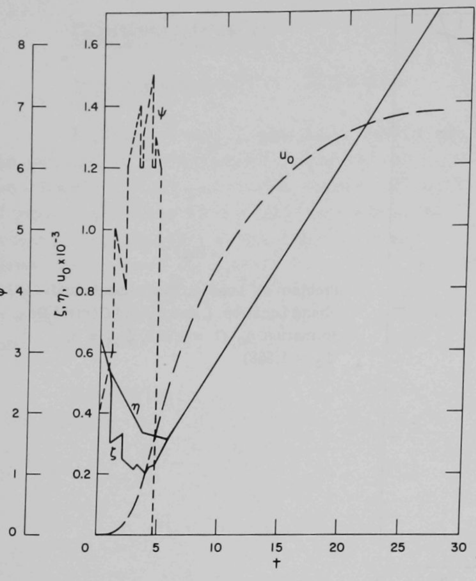



Fig. 8

Problem b: Load $\psi$, Hinge-circle and -band Location $\zeta$ and $\eta$, and Central Deformation $u_o$ ($i = 27$, $\psi_M = 6$, $\psi_e = 3.627$)

113-1879




113-1878

Fig. 9

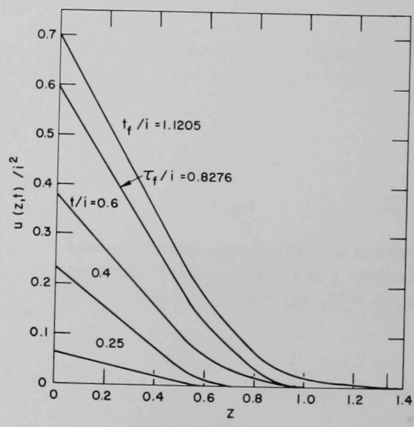
Problem c: Load $\psi$, Hinge-circle and -band Location $\zeta$ and $\eta$, and Central Deformation $u_o$ ($i$ = 24.875, $\psi_M$ = 7.5, $\psi_e$ = 4.527)

Figures 10-12 show the radial deformation shapes as a function of dimensionless axial position at various times for problems a, b, and c, respectively. On the figures, $\tau_f$ is the time at which the pulse ends.

We have, from Eqs. 1 and 7, that

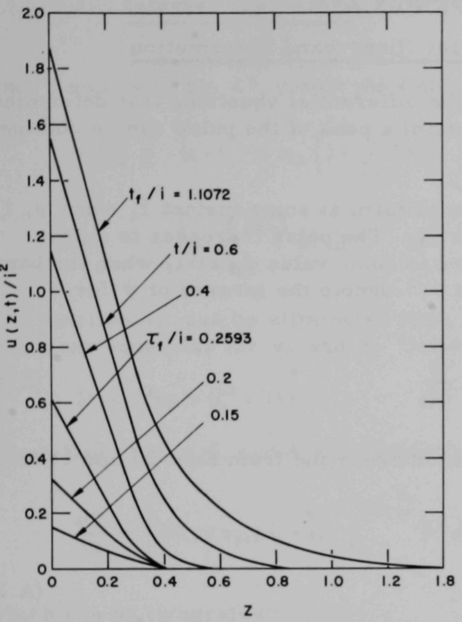
$$\left.\begin{aligned} \frac{t}{i} &= \frac{2\sigma_y H^{3/2}}{\sqrt{R}} \frac{T}{I}, \\ \frac{v}{i} &= 2\rho\sqrt{RH}\,\frac{V}{I}, \\ &\text{and} \\ \frac{u}{i^2} &= 4\rho\sigma_y H^2 \frac{U}{I^2}. \end{aligned}\right\} \tag{44}$$

Therefore, scaling the variables by the appropriate power of the impulse eliminates the arbitrary multiplicative factor in the time scale, represented by $T_0$ in the nondimensionalization and $\tau$ in the problem statements.



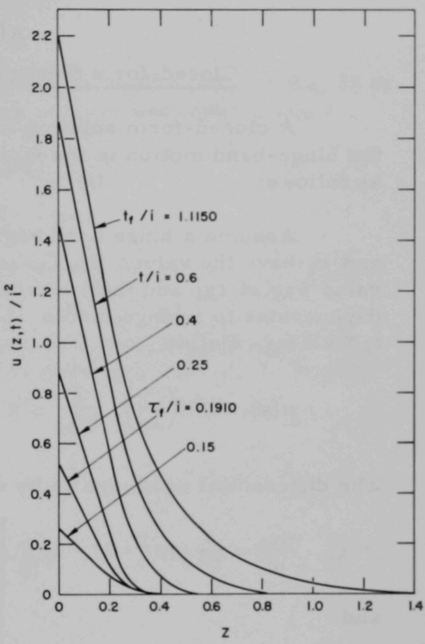

113-1884

Fig. 10. Problem a: Radial Deformation as a Function of Axial Position for Various Times ($t_y/i$ = 0.0690)




113-1888

Fig. 11. Problem b: Radial Deformation as a Function of Axial Position for Various Times ($t_y/i = 0$)




113-1889

Fig. 12. Problem c: Radial Deformation as a Function of Axial Position for Various Times ($t_y/i = 0$)

# APPENDIX A

## Closed-form Solution for Hinge-band Deformation

A closed-form solution to the differential equations that determine the hinge-band motion in the vicinity of a peak of the pulse can be obtained as follows:

Assume a hinge band begins to form at some instant $t_1$ when $\psi$, $\zeta$, and $v_0$ have the values $\psi_1$, $\zeta_1$, and $v_{01}$. The pulse increases to a peak value $\psi_M$ at $t_M$ and then decreases to some value $\psi_2$ at $t_2$ when the band degenerates to a hinge circle. Let $\psi^{-1}$ denote the inverse of $\psi$ for $t_1 \leq t \leq t_M$. Define

$$\mu(x) = \psi^{-1}\left(\frac{3}{2x}\right), \quad \frac{3}{2\psi_M} \leq x \leq \frac{3}{2\psi_1}. \tag{A.1}$$

The differential equations to be solved are found from Eqs. 23 and 15 to be

$$\left.\begin{aligned} \frac{dv_0}{dt} &= \frac{1}{\zeta^2}(4\psi\zeta - 3) - 1, \\ &\text{and} \\ \frac{d}{dt}\left(\frac{v_0 - v_\zeta}{\zeta}\right) &= \frac{6}{\zeta^3}(\psi\zeta - 1). \end{aligned}\right\} \tag{A.2}$$

### 1. Interval $t_1 \leq t \leq t_M$

Since $\zeta$ is related to $\psi$ through Eq. 21 in this interval, $\zeta$ can be eliminated from Eqs. A.2, resulting in simple linear differential equations for $v_0$ and $v_\zeta$; their solution is

$$\left.\begin{aligned} v_0(t) &= v_{01} - (t - t_1) + \frac{4}{3}\int_{t_1}^{t} \psi^2\, dt, \\ &\text{and} \\ v_\zeta(t) &= v_0(t) - \frac{1}{\psi(t)}\left[\psi_1 v_{01} + \frac{4}{3}\int_{t_1}^{t} \psi^3\, dt\right], \end{aligned}\right\} \tag{A.3}$$

making use of the initial conditions at $t_1$. Since $\zeta$ and $v_\zeta$ are known from Eqs. 21 and A.3, $\Omega$ is found from Eqs. 27 to be

$$\Omega(z) = t_1 + v_{01}\left(1 - \frac{z}{\zeta_1}\right) + \frac{4}{3}\int_{t_1}^{\mu(z)} \psi^2\, dt - \frac{8z}{9}\int_{t_1}^{\mu(z)} \psi^3\, dt. \tag{A.4}$$

A transcendental equation for $\eta$ is then obtained from Eqs. 26 and A.4.

2. Interval $t_M < t \le t_2$

The function $\Omega$, as given by Eq. A.4, is still used with Eq. 26 to find $\eta$ and, with Eq. 27, yields the following relation between $v_\zeta$ and $\zeta$:

$$v_\zeta(t) = -(t - t_1) + v_{01}\left(1 - \frac{\zeta}{\zeta_1}\right) + \frac{4}{3}\int_{t_1}^{\mu(\zeta)} \psi^2 \, dt$$

$$- \frac{8\zeta}{9}\int_{t_1}^{\mu(\zeta)} \psi^3 \, dt. \tag{A.5}$$

The function $v_\zeta$ can be eliminated from Eqs. A.2, resulting in a pair of nonlinear equations for $v_0$ and $\zeta$. Define an auxiliary function F through

$$F = \zeta^2 v_0 + (\zeta^2 - 3)\, t. \tag{A.6}$$

From Eqs. A.2, A.5, and A.6 we arrive at

$$\frac{dF}{dt} = \left[2(t_1 + v_{01})\,\zeta + \frac{8\zeta}{3}\int_{t_1}^{\mu(\zeta)} \psi^2 \, dt\right]\frac{d\zeta}{dt}, \tag{A.7}$$

which can be integrated to give

$$F = \zeta^2 (t_1 + v_{01}) - 3\mu(\zeta) + \frac{4\zeta^2}{3}\int_{t_1}^{\mu(\zeta)} \psi^2 \, dt + C. \tag{A.8}$$

Solving Eq. A.6 for $v_0$ and using Eq. A.8 with the continuity conditions at $t_M$, we obtain

$$v_0(t) = v_{01} - (t - t_1) + \frac{3}{\zeta^2}[t - \mu(\zeta)] + \frac{4}{3}\int_{t_1}^{\mu(\zeta)} \psi^2 \, dt. \tag{A.9}$$

The substitution of Eq. A.9 into the first of Eqs. A.2 then gives the following differential equation for $\zeta$:

$$\frac{3}{\zeta^2}[t - \mu(\zeta)]\frac{d\zeta}{dt} + 2\psi - \frac{3}{\zeta} = 0. \tag{A.10}$$

Equation A.10 can be written as a perfect differential having the standard form

$$X(\zeta,t)\, d\zeta + Y(\zeta,t)\, dt = 0, \tag{A.11}$$

with

$$X = \frac{3}{\zeta^2}[t - \mu(\zeta)], \quad Y = 2\psi(t) - \frac{3}{\zeta}, \text{ and } \frac{\partial X}{\partial t} = \frac{\partial Y}{\partial \zeta}. \tag{A.12}$$

Consequently, following the usual integration procedure and applying the continuity requirements at $t_M$, we arrive at the transcendental equation relating $\zeta$ and t,

$$3[\mu(\zeta) - t] + 2\zeta \int_{\mu(\zeta)}^{t} \psi \, dt = 0. \tag{A.13}$$

# APPENDIX B

## Local Series Solution of Differential Equations

The computer program RINGLOAD uses series solutions to the coupled nonlinear differential equations (Eqs. 16) in the vicinity of $t_y$ and $t_f$, since $v_0$ vanishes at these times and the second of Eqs. 16 becomes indeterminate. The series solution at $t = t_y$ is used to start the computer computation by calculating results at $t = t_y + \delta t$. The subroutine DIFSUB, which uses the Bulirsch-Stoer extrapolation method, is used to carry the computation forward until either the pulse ends and the closed-form solution (Eqs. 31) is applicable or the velocity $v_0$ becomes small. In the latter case, a series solution is then used to bring the calculation up to the final time $t_f$ at which the motion stops.

Consider a time $t^*$ when all the dependent variables are known quantities, and let $t^* + \delta t$ be a time when values of the dependent variables are desired. Assume $\delta t$ is small enough that power series solutions of Eqs. 16 in $\delta t$ are convergent. Define the auxiliary function $v_{aux}$ by

$$v_{aux}(t) = \frac{v_0(t)}{\zeta(t)}, \tag{B.1}$$

so that

$$v(z,t) = v_0(t) - z v_{aux}(t), \qquad 0 \le z \le \zeta(t). \tag{B.2}$$

Let

$$\left.\begin{aligned}
&\psi(t^* + \delta t) \approx \sum_{j=0}^{n} p_j (\delta t)^j, \\
&\zeta(t^* + \delta t) \approx \sum_{j=0}^{n} a_j (\delta t)^j, \\
&v_0(t^* + \delta t) \approx \sum_{j=0}^{n} b_j (\delta t)^j, \\
&\text{and} \\
&v_{aux}(t^* + \delta t) \approx \sum_{j=0}^{n} c_j (\delta t)^j,
\end{aligned}\right\} \tag{B.3}$$

where

$$p_0 = \psi(t^*), \; a_0 = \zeta(t^*), \; b_0 = v_0(t^*), \text{ and } c_0 = v_{aux}(t^*) = \frac{b_0}{a_0}. \tag{B.4}$$

From Eqs. B.1 and B.3 and the third of Eqs. 16, the series for the radial displacement is then given by

$$\left.\begin{aligned} u(z,t^*+\delta t) &= u(z,t^*) + \sum_{j=0}^{n} (b_j - zc_j)\frac{(\delta t)^{j+1}}{j+1}, \\ &\quad 0 \le z \le \zeta(t^*+\delta t), \\ \text{and} \\ u(z,t^*+\delta t) &= u(z,t^*), \quad z > \zeta(t^*+\delta t). \end{aligned}\right\} \tag{B.5}$$

The series for $\psi$ is assumed to be known from the given load-time function. From Eqs. B.1 and B.3, the coefficients of the series for $v_{aux}$ are given in terms of the coefficients of the series for $\zeta$ and $v_0$ by

$$c_n = \frac{1}{a_0}\left(b_n - \sum_{j=1}^{n} a_j c_{n-j}\right). \tag{B.6}$$

Substituting from Eqs. B.3 for $\psi$, $\zeta$, and $v_0$ into the first of Eqs. 16 and equating coefficients of like powers of $\delta t$, we obtain the following relations:

$$\left.\begin{aligned} & a_0b_0a_1 + a_0^2 + 2a_0p_0 - 3 = 0, \\ & 2a_0b_0a_2 + b_0a_1^2 + a_0a_1b_1 + 2a_0a_1 + 2a_1p_0 + 2a_0p_1 = 0, \\ & 3a_0b_0a_3 + 3b_0a_1a_2 + 2a_0b_1a_2 + a_1^2b_1 + a_0a_1b_2 + 2a_0a_2 \\ & \quad + a_1^2 + 2a_2p_0 + 2a_1p_1 + 2a_0p_2 = 0, \\ & 4a_0b_0a_4 + 4b_0a_1a_3 + 3a_0b_1a_3 + 2b_0a_2^2 + 3a_1b_1a_2 + 2a_0a_2b_2 \\ & \quad + a_1^2b_2 + a_0a_1b_3 + 2a_0a_3 + 2a_1a_2 + 2a_3p_0 + 2a_2p_1 \\ & \quad + 2a_1p_2 + 2a_0p_3 = 0, \\ \text{and} \\ & 5a_0b_0a_5 + 5b_0a_1a_4 + 4a_0b_1a_4 + 5b_0a_2a_3 + 4a_1b_1a_3 + 3a_0b_2a_3 \\ & + 2b_1a_2^2 + 3a_1a_2b_2 + 2a_0a_2b_3 + a_1^2b_3 + a_0a_1b_4 + 2a_0a_4 \\ & + 2a_1a_3 + a_2^2 + 2a_4p_0 + 2a_3p_1 + 2a_2p_2 + 2a_1p_3 + 2a_0p_4 = 0. \end{aligned}\right\} \tag{B.7}$$

The application of the same procedure to the second of Eqs. 16 gives the following relations:

$$\left.\begin{aligned}
&a_0^2b_1 + a_0^2 - 4a_0p_0 + 3 = 0,\\
&a_0^2b_2 + a_0a_1b_1 + a_0a_1 - 2a_1p_0 - 2a_0p_1 = 0,\\
&3a_0^2b_3 + 4a_0a_1b_2 + 2a_0b_1a_2 + a_1^2b_1 + 2a_0a_2 + a_1^2 - 4a_2p_0\\
&\qquad - 4a_1p_1 - 4a_0p_2 = 0,\\
&2a_0^2b_4 + 3a_0a_1b_3 + 2a_0a_2b_2 + a_1^2b_2 + a_0b_1a_3 + a_1b_1a_2 + a_0a_3\\
&\qquad + a_1a_2 - 2a_3p_0 - 2a_2p_1 - 2a_1p_2 - 2a_0p_3 = 0,\\
&\text{and}\\
&5a_0^2b_5 + 8a_0a_1b_4 + 6a_0a_2b_3 + 3a_1^2b_3 + 4a_0a_3b_2 + 4a_1a_2b_2\\
&\qquad + 2a_0b_1a_4 + 2a_1b_1a_3 + b_1a_2^2 + 2a_0a_4 + 2a_1a_3 + 2a_2^2\\
&\qquad - 4a_4p_0 - 4a_3p_1 - 4a_2p_2 - 4a_1p_3 - 4a_0p_4 = 0.
\end{aligned}\right\} \tag{B.8}$$

At the beginning of the deformation, when $t^* = t_y$, the shell is at rest, so that $v_0(t_y) = 0$. If $\psi$ passes gradually through the yield value of unity, the initial hinge-circle location is $\zeta(t_y) = 1$. However, if $\psi$ jumps instantaneously through yield to a value $\psi_M$ greater than unity, then $\zeta(t_y) = z_0$ given by Eqs. 30. Consequently, the two possible initial cases for $t^* = t_y$ are

$$b_0 = c_0 = 0,\ a_0 = p_0 = 1, \tag{B.9}$$

and

$$b_0 = c_0 = 0,\ p_0 = \psi_M,\ a_0 = \sqrt{\psi_M^2 + 3} - \psi_M. \tag{B.10}$$

The remaining $a_j$ and $b_j$ coefficients are found from Eqs. B.7 and B.8 to be

$$\left.\begin{aligned}
b_1 &= \frac{-1}{a_0^2}(a_0^2 - 4a_0p_0 + 3),\\
a_1 &= \frac{2a_0p_1}{k_1},\\
b_2 &= \frac{-1}{a_0^2}[a_0a_1(b_1 + 1) - 2(a_1p_0 + a_0p_1)],
\end{aligned}\right\} \text{(B.11)(Contd.)}$$

$$
\left.
\begin{aligned}
a_2 &= \frac{1}{k_2}[a_0a_1b_2 + a_1^2(b_1+1) + 2(a_1p_1 + a_0p_2)], \\
b_3 &= \frac{-1}{3a_0^2}[4a_0a_1b_2 + (2a_0a_2 + a_1^2)(b_1+1) - 4(a_2p_0 + a_1p_1 + a_0p_2)], \\
a_3 &= \frac{1}{k_3}[a_0a_1b_3 + (2a_0a_2 + a_1^2)\, b_2 + a_1a_2(3b_1+2) \\
&\quad + 2(a_2p_1 + a_1p_2 + a_0p_3)], \\
b_4 &= \frac{-1}{2a_0^2}[3a_0a_1b_3 + (2a_0a_2 + a_1^2)\, b_2 + (a_0a_3 + a_1a_2)(b_1+1) \\
&\quad - 2(a_3p_0 + a_2p_1 + a_1p_2 + a_0p_3)], \\
a_4 &= \frac{1}{k_4}[a_0a_1b_4 + (2a_0a_2 + a_1^2)\, b_3 + 3(a_0a_3 + a_1a_2)\, b_2 \\
&\quad + (2a_1a_3 + a_2^2)(2b_1+1) + 2(a_3p_1 + a_2p_2 + a_1p_3 + a_0p_4)], \\
&\text{and} \\
b_5 &= \frac{-1}{5a_0^2}[8a_0a_1b_4 + 3(2a_0a_2 + a_1^2)\, b_3 + 4(a_0a_3 + a_1a_2)\, b_2 \\
&\quad + (2a_0a_4 + 2a_1a_3 + a_2^2)(b_1+1) \\
&\quad - 4(a_4p_0 + a_3p_1 + a_2p_2 + a_1p_3 + a_0p_4)],
\end{aligned}
\right\}
\begin{array}{r}\text{(Contd.)}\\ \text{(B.11)}\end{array}
$$

where

$$
k_n = \frac{1}{a_0}[a_0^2(n-2) - 2a_0p_0(2n+1) + 3n]. \tag{B.12}
$$

If the initial conditions are given by Eqs. B.9, the first of Eqs. B.11 yields $b_1 = 0$, so that the $(\delta t)^2$ terms are the first nonvanishing terms in the power series for the velocity. Consequently, the numerical integration of the second of Eqs. 16 is initially rather slowly convergent if $\psi$ gradually passes through the yield value. On the other hand, $b_1 \neq 0$ for the initial conditions (Eqs. B.10), so that problems with an instantaneous jump at $t_y$ are easier to integrate numerically near $t_y$. This difference in the rapidity of convergence near $t_y$ shows up as shorter running times for problems with an initial jump.

If the shell is in motion at $t^*$, then $b_0 \neq 0$ and the five equations in each of Eqs. B.7 and B.8 are easily solved for $a_1, \ldots, a_5$ and $b_1, \ldots, b_5$, respectively, in terms of values of the functions at $t^*$.

In extrapolating to $t = t_f$, we use the numerical integration scheme until $v_0$ becomes small compared to its maximum during the deformation. Let $t^*$ be the time at which this small value is attained and at which the power-series coefficients are evaluated. Let

$$\left.\begin{aligned} t_f &= t^* + \delta t, \\ \text{and} \quad v_0(t_f) &= 0 \approx \sum_{j=0}^{n} b_j(\delta t)^j, \end{aligned}\right\} \tag{B.13}$$

where $\delta t$ is to be determined. We can approximate $\delta t$ by

$$\delta t \approx -\frac{2b_0}{b_1 + \sqrt{b_1^2 - 4b_0 b_2}}. \tag{B.14}$$

The value of $v_0$ at $t^* + \delta t$ is found from the power series, and if less than some selected constant $\epsilon_v$, the computation is concluded. If $v_0$ is greater than $\epsilon_v$, the extrapolation procedure is repeated starting from the new value.

## APPENDIX C

### Solution Behavior in the Vicinity of Instantaneous Jumps in $\psi$

In the vicinity of instantaneous jumps in $\psi$, there is the possibility of corresponding instantaneous jumps in the values of some of the dependent variables. It would be advantageous then to use the numerical-integration scheme up to the jump, make the appropriate change in the values of the variables, and continue on with the numerical integration, rather than trying to integrate through the jump, where the convergence of the solution may be poor. A study of the local power-series solution for the problem reveals that the dependent variables are continuous functions if no hinge-band formation occurs. If a hinge band already exists or is produced by the jump in $\psi$, then a corresponding jump in $\zeta$ occurs.

At $t = t^*$, let $\psi$ be such that

$$\begin{aligned} \psi(t^* + \delta t) &= p_0 + p_1 \delta t \\ &= \psi(t^*) + \delta\psi. \end{aligned} \tag{C.1}$$

We wish to determine what happens to the solution of the differential equations as $\delta t \to 0$ with $\delta\psi$ kept constant.

Consider first the case when no hinge band exists and Eqs. 16 are appropriate. The series solutions for $\zeta$ and $v_0$ are given in Appendix B. By Eqs. B.7 and B.8, we can write

$$\left.\begin{aligned} a_n &= a_{n0}p_0 + a_{n1}p_1, \\ &\text{and} \\ b_n &= b_{n0}p_0 + b_{n1}p_1, \end{aligned}\right\} \tag{C.2}$$

where $a_{n0}$, $a_{n1}$, $b_{n0}$, and $b_{n1}$ are independent of $p_0$ and $p_1$. From Eqs. B.7 and B.8,

$$a_{01} = a_{11} = b_{01} = b_{11} = 0, \tag{C.3}$$

so that the series for $\zeta$ and $v_0$ can be written

$$\zeta(t^* + \delta t) = \zeta(t^*) + a_{10}\delta t + \sum_{j=2}^{n} (a_{j0}p_0 + a_{j1}p_1)(\delta t)^j,$$

and

$$v_0(t^* + \delta t) = v_0(t^*) + b_{10}\delta t + \sum_{j=2}^{n} (b_{j0}p_0 + b_{j1}p_1)(\delta t)^j. \tag{C.4}$$

It is apparent then that

$$\zeta(t^* + \delta t) \to \zeta(t^*), \text{ and } v_0(t^* + \delta t) \to v_0(t^*) \text{ as } \delta t \to 0, \qquad (C.5)$$

keeping $p_1 \delta t$ constant. It is assumed in this analysis that

$$[\psi(t^*) + \delta\psi]\ \zeta(t^* + \delta t) \leq 3/2. \qquad (C.6)$$

If Inequality C.6 does not hold, that is, if the instantaneous jump produces a hinge band, the instantaneous jump may be decomposed into two parts by

$$\left.\begin{aligned} \psi(t^* + \delta t) &= p_0 + p_1' \delta t' + p_1'' \delta t'', \\ \text{and} \qquad & \\ \delta t &= \delta t' + \delta t'', \end{aligned}\right\} \qquad (C.7)$$

with $\delta t'$ determined from

$$(p_0 + p_1 \delta t')\ \zeta(t^*) = 3/2. \qquad (C.8)$$

Letting $\delta t' \to 0$ carries the deformation up to the point of producing a hinge band. The remainder of the instantaneous jump, produced by letting $\delta t'' \to 0$ with $p_1'' \delta t''$ kept constant, then occurs while a hinge band exists. This latter case will be considered next.

The governing differential equations when a hinge band exists are Eqs. A.2. If $\psi$ is increasing, $\zeta$ is found from Eq. 21 and $\Omega$ is determined from Eq. 27; if $\psi$ is decreasing, $\Omega$ is now a known function and Eq. 27 gives a relation between $\zeta$ and $v_\zeta$. Define $v_{aux}$ now by

$$v_{aux}(t) = \frac{v_0(t) - v_\zeta(t)}{\zeta(t)}, \qquad (C.9)$$

and, as in Eq. C.1, let a jump of $\delta\psi$ occur in $\psi$ at $t = t^*$. Employing a similar series analysis to that used previously, we find for $\delta\psi$ positive that $v_0$, $v_{aux}$, and $\eta$ are continuous at $t^*$, while $\zeta$ jumps to $3/\{2[\psi(t^*) + \delta\psi]\}$ and the corresponding jump in $v_\zeta$ is found from Eq. C.9. The portion of $\Omega$ generated by the jump in $\psi$ is given by

$$\left.\begin{aligned} \Omega(z) &= v_0(t^*) - v_{aux}(t^*)\, z + t^*, \\ \text{for} \qquad & \\ \zeta(t^*) &\geq z \geq \frac{3}{2[\psi(t^*) + \delta\psi]}. \end{aligned}\right\} \qquad (C.10)$$

On the other hand, if $\delta\psi$ is negative, then $v_0$, $v_{aux}$, $v_\zeta$, $\zeta$, and $\eta$ are continuous at $t^*$.

Since $\eta$, the outer edge of the hinge band, is always a continuous function, the length of the region of plastic deformation is also a continuous function of time and does not vary abruptly, even with severe load changes.

# APPENDIX D

## Computer Programs

The dynamic plastic deformation of a tube acted on by a ring load of arbitrary pulse shape is computed by the CDC-3600 program RINGLOAD and its auxiliary subroutines DIFSUB,[6] DIFFUN, OMEGA, OMEGINV, ZETAEQ, PULSEINF, PSIFUN, and PSICOEF. The subroutines PULSEINF, PSIFUN, and PSICOEF are specialized to the particular form of the pressure pulse; RINGLOAD and the other subroutines do not depend on the pulse shape. Two versions of the set PULSEINF, PSIFUN, and PSICOEF are given here. The Piecewise Linear Version treats pulses that are specified by a number of $t, \psi$ pairs and uses linear interpolation to obtain values of $\psi$ for times between the prescribed points. This version can be used to handle any pulse shape by specifying a sufficient number of data points, and its use is recommended for most problems. If, however, a parameter study is to be made for a number of problems involving the same general pulse form and this form is tedious to approximate by a piecewise-linear function, it may be advantageous to write special subroutines for the pulse form. This was done for the case of a pulse that rises linearly to its maximum and then decays exponentially, that is,†

$$\left.\begin{aligned} \psi &= \psi_M \frac{(t - \tau_0)}{(\tau_M - \tau_0)}, & \tau_0 &\le t \le \tau_M; \\ &= \psi_M e^{-(t - \tau_M)/\tau}, & \tau_M &< t \le \tau_F; \\ &= 0, & t &> \tau_F. \end{aligned}\right\} \qquad (D.1)$$

The subroutines PULSEINF, PSIFUN, and PSICOEF for a pulse defined by Eqs. D.1 are given as the Exponential Decay Version. Problems of this type can also be handled by the Piecewise Linear Version if an adequate number of points on the exponential decay portion of the pulse are used as input data.

The input needed for a computation consists of two parts: input for the program RINGLOAD and input for the subroutine PULSEINF.

### 1. Input for RINGLOAD

Card 1. FORMAT (10A8)

Problem name or identifier (Anything may be punched on this card and will be printed at the beginning of the output.)

† The quantities $\tau_0$ and $\tau_M$ may be made equal to produce an instantaneous jump to the pulse maximum.

Card 2. FORMAT (I2, 2X, I2, 2X, I2)

ISCALE: If zero, results are not scaled. If one, results are scaled by the appropriate power of the impulse i, as explained below.

NTSINT: $\Delta t/i = 1/\text{NTSINT}$ is the scaled time interval at which the output is printed. NTSINT must be at least 10; 20 is a convenient value for most problems.

NZINT: $\Delta z = 1/\text{NZINT}$ is the axial interval at which radial displacements are printed. NZINT must not exceed 25; 10 or 20 gives an adequate number of points for a graph. If NZINT is zero, then $z = 0$ is the only location at which the displacement is computed.

2. Input for PULSEINF. Piecewise Linear Version

Card 3. FORMAT (I2)

NJ: Number of input data points. NJ must not exceed 99.

Cards 4, 5, 6, ..., NJ + 3 FORMAT (2E15.7)

TJ, PSIJ: Input data pairs $t_j$, $\psi_j$.

3. Input for PULSEINF. Exponential Decay Version

Card 3. FORMAT (5E15.7)

TAUO, TAUM, TAUF, TAU, PSIM: Parameters in puls shape, $\tau_0$, $\tau_M$, $\tau_F$, $\tau$, $\psi_M$, as in Eqs. D.1.

The printed output of the program includes first some general parameters associated with the pulse shape, such as $t_y$, $t_m$, i, $\psi_e$ and the total area under the pulse curve. The values of $t_m$, i, and $\psi_e$ that are printed are based on integrals of the pulse from $t_y$ to the end of the pulse. If the motion stops before the end of the pulse,† it is advisable to rerun the problem, chopping off the pulse input values at approximately $t_f$ as given by the first run;†† then the integrals will be taken over the interval $t_y$ to $t_f$ as required in Eqs. 7. For many problems in which the motion stops before the end of the pulse, the pulse has decayed enough that the difference in the values of $t_m$, i, and $\psi_e$ between integrating to the end of the pulse and integrating to the end of the motion is negligible.

† A statement announcing this is printed at the end of the output.

†† If the pulse data input is cut off at exactly $t_f$, there will be convergence problems in the final time step of the second problem run. Chopping at a time value equal to $t_f$ to the first few significant figures avoids this difficulty and is accurate enough for the parameter evaluations.

The output that gives the history of the deformation is $t$, $\psi(t)$, $u_0(t)$, $v_0(t)$, $\zeta(t)$, $\eta(t)$, NOFNS, and $u(n\Delta z,t)$, $n = 1, 2, 3, \ldots$, if ISCALE = 0; the printouts are at intervals of $\Delta t$ plus intermediate times when results are calculated for program requirements. The quantity NOFNS is the number of evaluations of the right sides of the differential equations and is an indication of the difficulty of convergence of the numerical integration procedure.

If ISCALE = 1, the quantities printed are $t/i$, $\psi$, $u_0/i^2$, $v_0/i$, $\zeta(t)$, $\eta(t)$, NOFNS, and $u(n\Delta z,t)/i^2$. The scaling by powers of $i$ is such that the arbitrary time constant $T_0$ appearing in Eqs. 1 is eliminated, as well as the arbitrary factor in the time scale of the pulse, as indicated in Eqs. 44.

From Eqs. 16 we see that $\partial u/\partial t$ is found differently, depending on whether $z$ is greater or less than $\zeta$. In RINGLOAD, if a time step carries $\zeta$ past an axial position $z$ where the deformation is desired, the program interpolates back to find the time step for which $\zeta$ and this $z$-location coincide; then each $z$ at which the deformation is computed is either inside or outside of $\zeta$ for the time step, and Eqs. 16 can be applied.

The function $\Omega(t)$ mapped out through the use of Eq. 27 is stored as a pair of vectors ZL(JZ), OMEGAL(JZ). Linear interpolation is used to obtain values intermediate to those contained in the lists; that is, $\Omega$ is assumed to be a linear function of $z$ between the stored values.

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      PROGRAM RINGLOAD
C
C     DYNAMIC PLASTIC DEFORMATION OF A CIRCULAR CYLINDRICAL SHELL
C     PRODUCED BY A RING LOAD WITH AN ARBITRARY PULSE SHAPE, PSI(T).
C
C     AUXILIARY PROGRAMS REQUIRED ARE ANL LIBRARY SUBROUTINE DIFSUB
C     AND SPECIAL SUBROUTINES PULSEINF, PSIFUN, PSICOEF, DIFFUN,
C     OMEGA, OMEGINV, AND ZETAEQ.
C
C     RINGLOAD, DIFSUB, DIFFUN, OMEGA, OMEGINV, AND ZETAEQ ARE
C     GENERAL PROGRAMS APPROPRIATE TO ANY PULSE SHAPE, WHILE
C     PULSEINF, PSIFUN, AND PSICOEF MUST BE WRITTEN FOR THE
C     PARTICULAR FORM OF THE PULSE.  GENERAL PURPOSE VERSIONS OF
C     PULSEINF, PSIFUN, AND PSICOEF HAVE BEEN PROGRAMMED FOR THE
C     CASE WHERE PULSE VALUES ARE PRESCRIBED AT A SUFFICIENT NUMBER
C     OF TIME POINTS SUCH THAT LINEAR INTERPOLATION BETWEEN VALUES
C     IS PERMISSIBLE.
C
C     REQUIRED INPUT FROM DATA CARDS.
C         CARD 1.    FORMAT(10A8)
C     NPROB -- PROBLEM NAME OR IDENTIFIER.
C         CARD 2.    FORMAT(I2,2X,I2,2X,I2)
C     ISCALE -- IF ZERO, RESULTS ARE NOT SCALED. IF ONE, RESULTS
C     ARE SCALED BY THE APPROPRIATE POWER OF THE PLASTIC IMPULSE.
C     NTSINT -- 1/NTSINT IS THE SCALED TIME INTERVAL AT WHICH
C     OUTPUT IS DESIRED.  NTSINT MUST BE .GE. 10.
C     NZINT -- 1/NZINT IS THE AXIAL INTERVAL AT WHICH DISPLACEMENTS
C     ARE DESIRED.  NZINT MUST BE .LE.25.  IF NZINT=0, THEN Z=0 IS
C     THE ONLY LOCATION AT WHICH THE DISPLACEMENT IS COMPUTED.
C
C     PLUS INPUT REQUIRED FOR SUBROUTINE PULSEINF.
C
C     PRINTED OUTPUT
C     T -- TIME.
C     PSI(T) -- PULSE VALUE.
C     U(0,T) -- RADIAL DISPLACEMENT AT Z=0.
C     V(0,T) -- RADIAL VELOCITY AT Z=0.
C     ZETA(T) -- LOCATION OF HINGE CIRCLE OR OUTER EDGE OF HINGE BAND.
C     ETA(T) -- LOCATION OF INNER EDGE OF HINGE BAND.
C     NOFNS -- NUMBER OF USES OF SUBROUTINE DIFFUN.  (GIVES AN
C     INDICATION OF THE RAPIDITY OF CONVERGENCE OF THE SOLUTION.)
C     U(Z,T) -- RADIAL DISPLACEMENT AT Z=N/NZINT, N=1,2,3,....
C
C     IN THE BODY OF THE PROGRAM, PHASE 1 REFERS TO DEFORMATION
C     WITH A HINGE CIRCLE AT ZETA, WHILE PHASE 2 REFERS TO
C     DEFORMATION WITH A HINGE BAND BETWEEN ZETA AND ETA.
C
      DIMENSION TMAX(50),PSIMAX(50),TMAXS(50),U(50),US(50)
      DIMENSION F(4),DF(4),FMAX(4),FF(4),RLTVER(4),NPROB(10)
      DIMENSION TJMP(50),PSIJMP(50),TJMPS(50),UF(50),SPSI(4)
      COMMON/COMDIF/NOFNS,KPHASE,TAUJMP,PJMP
      COMMON/COMOMEG/JOM,OMEGAL(1001),ZL(1001)
    1 READ 170,NPROB
      IF(EOF,60)2,3
    2 STOP
    3 PRINT 171,NPROB
      READ 172,ISCALE,NTSINT,NZINT
      CALL PULSEINF(TO,TF,TYIELD,TOTIMP,PLASIMP,TMEAN,NOMAX,TMAX,
     1     PSIMAX,INSTJMP,NOJMP,TJMP,PSIJMP,KSTOP)
      IF(KSTOP)4,5
    4 PRINT 173,KSTOP
      GO TO 1
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C******** DATA INPUT, PARAMETER DETERMINATIONS, AND ASSOCIATED PRINTOUTS
    5 IF(INSTJMP)6,7
    6 PRINT 174,INSTJMP
    7 EFFLOAD=PLASIMP/(2.0*TMEAN)
      PRINT 175,TO,TF,TYIELD,TOTIMP,PLASIMP,TMEAN,EFFLOAD
    8 TOS=TO/PLASIMP
      TFS=TF/PLASIMP
      TYIELDS=TYIELD/PLASIMP
      TMEANS=TMEAN/PLASIMP
      PRINT 176,TOS,TFS,TYIELDS,TMEANS
    9 DO 10 K=1,NOMAX
   10 TMAXS(K)=TMAX(K)/PLASIMP
      PRINT 177,(K,TMAX(K),PSIMAX(K),TMAXS(K),K=1,NOMAX)
      IF(NOJMP.EQ.0) GO TO 16
      DO 15 K=1,NOJMP
   15 TJMPS(K)=TJMP(K)/PLASIMP
      PRINT 180,(K,TJMP(K),PSIJMP(K),TJMPS(K),K=1,NOJMP)
   16 CONTINUE
      DTS=1.0/NTSINT
      DT=DTS*PLASIMP
      IF(NZINT)11,12
   11 DZ=1.0/NZINT
      GO TO 13
   12 DZ=0.0
   13 PRINT 178,DT,DTS,DZ
C**** CHECK SIZE OF SELECTED DTS
      IF(NTSINT.GE.10) GO TO 17
      PRINT 179
      GO TO 1
C**** CHECK SIZE OF SELECTED DZ
   17 IF(NZINT.LE.25) GO TO 18
      PRINT 181
      GO TO 1
C**** PRINT OUTPUT HEADINGS
   18 IF(ISCALE)19,20
   19 PRINT 182
      IF(NZINT) PRINT 183
      GO TO 21
   20 PRINT 184
      IF(NZINT) PRINT 185
   21 CONTINUE
C******** INITIALIZATION OF U(Z,T)
      NZTOT=1.73205*NZINT
      DO 22 IZ=1,NZTOT
      U(IZ)=0.C
      US(IZ)=0.0
   22 CONTINUE
C******** INITIAL VALUES AT TYIELD
      UO=UOS=VC=VOS=UAUX=VAUX=0.0
      NOFNS=0
      PSIO=PSIFUN(TO)
      ZETAO=1.0
      IF(ISCALE) GO TO 23
      PRINT 187,TO,PSIO,UO,VO,ZETAO,NOFNS
      GO TO 24
   23 PRINT 187,TOS,PSIO,UOS,VOS,ZETAO,NOFNS
   24 PSI=PSIFUN(TYIELD)
      IF(INSTJMP) GO TO 25
      ZETA=1.0
      GO TO 26
   25 ZETA=SQRTF(3.0+PSI**2)-PSI
   26 IF(ISCALE) GO TO 27
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      PRINT 187,TYIELD,PSI,UO,VO,ZETA,NOFNS
      PRINT 190,(U(IZ),IZ=1,NZTOT)
      GO TO 28
   27 PRINT 187,TYIELDS,PSI,UOS,VOS,ZETA,NOFNS
      PRINT 190,(US(IZ),IZ=1,NZTOT)
   28 CONTINUE
C******** SOLUTION BETWEEN TYIELD AND TI, TI NEAR TYIELD
C**** USE POWER SERIES SOLUTION OF DIFFERENTIAL EQUATIONS NEAR TYIELD.
C**** CALCULATE COEFFICIENTS IN POWER SERIES EXPANSIONS.
C**** ZETA = A0 + A1*D + A2*D**2 +..., D = T-TYIELD
C**** VO = B0 + B1*D + B2*D**2 +...
C**** VAUX = C0 + C1*D + C2*D**2 +...
C**** PSI = P0 + P1*D + P2*D**2 +...
      CALL PSICOEF(TYIELD,SPSI)
      P0=PSI
      P1=SPSI(1)
      P2=SPSI(2)
      P3=SPSI(3)
      P4=SPSI(4)
      A0=ZETA
      B0=C0=0.0
      B1=-(A0**2+3.0-4.0*A0*P0)/A0**2
      C1=B1/A0
      A1=-2.0*A0*P1/(B1*A0+2.0*P0+2.0*A0)
      B2=-(B1*A1*A0-2.0*A1*P0-2.0*A0*P1+A1*A0)/A0**2
      C2=(B2-C1*A1)/A0
      A2=-(B2*A1*A0+B1*A1**2+2.0*A1*P1+2.0*A0*P2+A1**2)/(2.0*(B1*A0+P0
     1    +A0))
      B3=-(4.0*B2*A1*A0+2.0*B1*A2*A0+B1*A1**2-4.0*A2*P0-4.0*A1*P1
     1    -4.0*A0*P2+2.0*A2*A0+A1**2)/(3.0*A0**2)
      C3=(B3-C2*A1-C1*A2)/A0
      A3=-(B3*A1*A0+2.0*B2*A2*A0+B2*A1**2+3.0*B1*A2*A1+2.0*(A2*P1+A1*P2
     1    +A0*P3+A2*A1))/(3.0*B1*A0+2.0*P0+2.0*A0)
      B4=-(3.0*B3*A1*A0+2.0*B2*A2*A0+B2*A1**2+B1*A3*A0+B1*A2*A1
     1    -2.0*(A3*P0+A2*P1+A1*P2+A0*P3)+A3*A0+A2*A1)/(2.0*A0**2)
      C4=(B4-C3*A1-C2*A2-C1*A3)/A0
      A4=-(B4*A1*A0+B3*A1**2+2.0*B3*A2*A0+3.0*B2*A2*A1+3.0*B2*A3*A0
     1    +2.0*B1*A2**2+4.0*B1*A3*A1+2.0*(A3*P1+A2*P2+A1*P3+A0*P4)
     2    +2.0*A3*A1+A2**2)/(2.0*(2.0*B1*A0+P0+A0))
      B5=-(8.0*B4*A1*A0+6.0*B3*A2*A0+3.0*B3*A1**2+4.0*B2*A3*A0
     1    +4.0*B2*A2*A1+2.0*B1*A4*A0+2.0*B1*A3*A1+B1*A2**2-4.0*(A4*P0
     2    +A3*P1+A2*P2+A1*P3+A0*P4)+2.0*A4*A0+2.0*A3*A1+A2**2)/
     3    (5.0*A0**2)
      C5=(B5-C4*A1-C3*A2-C2*A3-C1*A4)/A0
C**** FIND TIME TI AT WHICH PSI DIFFERS FROM PSI AT TYIELD BY AT MOST
C****       THREE PERCENT.
      EPSSER=0.03
      DELT=0.01*TMEAN
   37 TI=TYIELD+DELT
      PSII=PSIFUN(TI)
      EPSI=ABSF(PSII/P0-1.0)
      IF(EPSI.LE.EPSSER) GO TO 38
      DELT=DELT*EPSSER/EPSI
      GO TO 37
C**** EVALUATE POWER SERIES AT TI
   38 ZETA=A0+A1*DELT+A2*DELT**2+A3*DELT**3+A4*DELT**4
      VO=B0+B1*DELT+B2*DELT**2+B3*DELT**3+B4*DELT**4+B5*DELT**5
      VAUX=C0+C1*DELT+C2*DELT**2+C3*DELT**3+C4*DELT**4+C5*DELT**5
      UO=UO+B0*DELT+B1*DELT**2/2.0+B2*DELT**3/3.0+B3*DELT**4/4.0
     1    +B4*DELT**5/5.0+B5*DELT**6/6.0
      UAUX=UAUX+C0*DELT+C1*DELT**2/2.0+C2*DELT**3/3.0+C3*DELT**4/4.0
     1    +C4*DELT**5/5.0+C5*DELT**6/6.0
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C**** PRINT RESULTS AT TI
      IF(ISCALE) GO TO 39
      PRINT 188,TI,PSII,UO,VO,ZETA,O
      GO TO 40
   39 TIS=TI/PLASIMP
      VOS=VO/PLASIMP
      UOS=UO/PLASIMP**2
      PRINT 188,TIS,PSII,UOS,VOS,ZETA,O
   40 IF(NZINT)41,50
C**** CALCULATION OF U(Z,TI)
   41 NZ=NZINT
   42 IF(NZ*DZ.LE.ZETA) GO TO 43
      NZ=NZ-1
      GO TO 42
   43 DO 44 IZ=1,NZ
   44 U(IZ)=UO-IZ*DZ*UAUX
      IF(ISCALE) GO TO 45
      PRINT 189,(U(IZ),IZ=1,NZTOT)
      GO TO 50
   45 DO 46 IZ=1,NZ
   46 US(IZ)=U(IZ)/PLASIMP**2
      PRINT 189,(US(IZ),IZ=1,NZTOT)
   50 CONTINUE
C******** SCHECULED PRINTOUTS AT TMAX(K), TJMP(K), TF, AND INTEGER
C******** MULTIPLES OF DT
      DO 51 JMAX=1,NOMAX
      IF(TMAX(JMAX).GT.TI) GO TO 52
   51 CONTINUE
      JMAX=NOMAX+1
   52 DO 53 JJMP=1,NOJMP
      IF(TJMP(JJMP).GT.TI) GO TO 54
   53 CONTINUE
      JJMP=NOJMP+1
   54 DO 55 IT=1,1000
      IF(IT*DT.GT.TI) GO TO 56
   55 CONTINUE
      GO TO 1
   56 CONTINUE
C******** SOLUTION BETWEEN TI AND TF
C**** USE SUBRCUTINES DIFSUB AND DIFFUN
C**** F(1)=VO, F(2)=ZETA, F(3)=UOINC, F(4)=UAINC
C**** INITIALIZATIONS FOR DIFSUB
      F(1)=VO
      F(2)=ZETA
      F(3)=F(4)=FF(3)=FF(4)=0.0
      NEQ=4
      NOFNS=0
      MORD=6
      METH=0
      KKSTP=+1
      KPHASE=0
      DTMIN=0.000001*PLASIMP
      EPSERR=0.00001
      EPSZ=0.00001
      EPSPR=0.000001
      EPSRV=0.001
      EPSV1=0.01
      EPSV2=0.000001
      NZZ=NZ
      DO 57 I=1,4
   57 FMAX(I)=F(I)
      T=TI
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C**** SELECTICN OF NEXT SCHEDULED PRINTOUT TIME TPR
   58 CONTINUE
      IF(JJMP.GT.NOJMP) 59,60
   59 TAUJMP=2.0*PLASIMP
      PJMP=0.0
      GO TO 61
   60 TAUJMP=TJMP(JJMP)
      PJMP=PSIJMP(JJMP)
   61 IF(JMAX.GT.NOMAX) 62,63
   62 TAUMAX=2.0*PLASIMP
      PMAX=0.0
      GO TO 64
   63 TAUMAX=TMAX(JMAX)
      PMAX=PSIMAX(JMAX)
   64 TDT=IT*CT
      TPR=AMIN1(TDT,TAUJMP,TAUMAX)
      IF(TPR.GT.TF) TPR=TF
C**** INTEGRATICN OF NONLINEAR DIFFERENTIAL EQUATIONS USING DIFSUB
   70 H=TPR-T
   71 TT=T
      FF(1)=F(1)
      FF(2)=F(2)
      F(3)=F(4)=0.0
      CALL DIFSUB(NEQ,T,F,DF,H,DTMIN,EPSERR,MORD,METH,FMAX,RLTVER,KKSTP)
      PSI=PSIFUN(T)
      IF(T.EQ.TAUJMP) PSI=PSI-PJMP
      IF(KKSTP.GE.0) 73,72
   72 PRINT 191,T
      GO TO 1
   73 VO=F(1)
      ZETA=F(2)
      UOINC=F(3)
      UAINC=F(4)
      TS=T/PLASIMP
      VOS=VO/PLASIMP
      IF(NZINT.EQ.C) GO TO 74
      IF(ZETA.LE.NZ*DZ) GO TO 90
      IF(ZETA.GE.(NZ+1)*DZ) GO TO 95
   74 IF(PSI*ZETA.GT.1.5) GO TO 122
      IF(VO.LT.0.0) GO TO 100
   75 UO=UO+UCINC
      UOS=UO/PLASIMP**2
      IF(NZINT) 76,78
   76 DO 77 IZ=1,NZ
      U(IZ)=U(IZ)+UOINC-IZ*DZ*UAINC
      US(IZ)=U(IZ)/PLASIMP**2
   77 CONTINUE
   78 NZ=NZZ
      IF(ABSF(TPR-T).LE.EPSPR) GO TO 82
C**** INTERMECIATE PRINTOUT
      IF(ISCALE) GO TO 80
      PRINT 188,T,PSI,UO,VO,ZETA,NCFNS
      PRINT 189,(U(IZ),IZ=1,NZTOT)
      GO TO 81
   80 PRINT 188,TS,PSI,UOS,VOS,ZETA,NOFNS
      PRINT 189,(US(IZ),IZ=1,NZTOT)
   81 IF(VO/FMAX(1).LE.EPSRV) 10C,70
C**** SCHEDULEC PRINTOUT
   82 IF(ISCALE) GO TO 83
      PRINT 187,T,PSI,UO,VO,ZETA,NCFNS
      PRINT 19C,(U(IZ),IZ=1,NZTOT)
      GO TO 85
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   83 PRINT 187,TS,PSI,UOS,VOS,ZETA,NOFNS
      PRINT 190,(US(IZ),IZ=1,NZTOT)
   85 IF(TPR.EC.TF) GO TO 110
      IF(VO/FMAX(1).LE.EPSRV) GO TO 100
      IF(TPR.EQ.TAUJMP.AND.(PSI+PJMP)*ZETA.GT.1.5) GO TO 222
      IF(TPR.EC.TAUJMP) JJMP=JJMP+1
      IF(TPR.EC.TAUMAX) JMAX=JMAX+1
      IF(TPR.EC.TCT) IT=IT+1
      GO TO 58
C**** ZETA HAS CROSSED NZ*DZ FROM RIGHT TO LEFT IN THIS TIME INTERVAL
   90 NZZ=NZ-1
   91 UOP=UO+UCINC
      UOPS=UOP/PLASIMP**2
      IF(ISCALE) GO TO 92
      PRINT 188,T,PSI,UOP,VO,ZETA,NOFNS
      GO TO 93
   92 PRINT 188,TS,PSI,UOPS,VOS,ZETA,NOFNS
   93 H=(T-TT)*(NZ*DZ-FF(2))/(ZETA-FF(2))
      DO 94 I=1,4
   94 F(I)=FF(I)
      T=TT
      CALL DIFSUB(NEQ,T,F,DF,H,DTMIN,EPSERR,MORD,METH,FMAX,RLTVER,KKSTP)
      PSI=PSIFUN(T)
      IF(T.EQ.TAUJMP) PSI=PSI-PJMP
      IF(KKSTP.LT.C) GO TO 72
      VO=F(1)
      ZETA=F(2)
      UOINC=F(3)
      UAINC=F(4)
      TS=T/PLASIMP
      VOS=VO/PLASIMP
      IF(ABSF(ZETA-NZ*DZ).LE.EPSZ) 74,91
C**** ZETA HAS CROSSED (NZ+1)*DZ FROM LEFT TO RIGHT IN THIS TIME INTERVAL
   95 NZZ=NZ+1
   96 UOP=UO+UCINC
      UOPS=UOP/PLASIMP**2
      IF(ISCALE) GO TO 97
      PRINT 188,T,PSI,UOP,VO,ZETA,NOFNS
      GO TO 98
   97 PRINT 188,TS,PSI,UOPS,VOS,ZETA,NOFNS
   98 H=(T-TT)*((NZ+1)*DZ-FF(2))/(ZETA-FF(2))
      DO 99 I=1,4
   99 F(I)=FF(I)
      T=TT
      CALL DIFSUB(NEQ,T,F,DF,H,DTMIN,EPSERR,MORD,METH,FMAX,RLTVER,KKSTP)
      PSI=PSIFUN(T)
      IF(T.EQ.TAUJMP) PSI=PSI-PJMP
      IF(KKSTP.LT.0) GO TO 72
      VO=F(1)
      ZETA=F(2)
      UOINC=F(3)
      UAINC=F(4)
      TS=T/PLASIMP
      VOS=VO/PLASIMP
      IF(ABSF(ZETA-(NZ+1)*DZ).LE.EPSZ) 74,96
C******** DEFCRMATION STOPS BEFORE TF
  100 IF(ABSF(VOS).LE.EPSV1) GO TO 101
C**** EXTRAPOLATE TO VO=C, APPROXIMATELY
      H=(T-TT)*VO/(FF(1)-VO)
      GO TO 71
  101 IF(ABSF(VOS).GT.EPSV2) GO TO 102
      PRINT 192
```

```
      GO TO 1
  102 CONTINUE
C**** USE POWER SERIES TO IMPROVE APPROXIMATION TO VO=0
C**** COEFFICIENTS
      CALL PSICOEF(T,SPSI)
      P0=PSI
      P1=SPSI(1)
      P2=SPSI(2)
      P3=SPSI(3)
      P4=SPSI(4)
      A0=ZETA
      B0=VO
      C0=B0/A0
      A1=(3.0-2.0*P0*A0-A0**2)/(A0*B0)
      B1=(4.0*P0*A0-A0**2-3.0)/A0**2
      C1=(B1-C0*A1)/A0
      A2=(B0*A1**2+B1*A1*A0+2.0*(PC*A1+P1*A0+A1*A0))/(-A0*B0)
      B2=-(B1*A1*A0-2.0*(P0*A1+P1*A0)+A1*A0)/A0**2
      C2=(B2-C1*A1-C0*A2)/A0
      A3=-(3.0*B0*A2*A1+2.0*B1*A2*A0+B1*A1**2+B2*A1*A0+2.0*(P0*A2+P1*A1
     1    +P2*A0+A2*A0)+A1**2)/(3.C*A0*B0)
      B3=-(4.0*B2*A1*A0+2.0*B1*A2*A0+B1*A1**2-4.0*(P0*A2+P1*A1+P2*A0)
     1    +2.C*A2*A0+A1**2)/(3.0*AC**2)
      C3=(B3-C2*A1-C1*A2-C0*A3)/A0
      A4=-(4.0*B0*A3*A1+3.0*B1*A3*A0+2.0*B0*A2**2+3.0*B1*A2*A1
     1    +2.0*B2*A2*A0+B2*A1**2+B3*A1*A0+2.0*(P0*A3+P1*A2+P2*A1+P3*A0
     2    +A3*AC+A2*A1))/(4.0*A0*BC)
      B4=-(3.C*B3*A1*A0+2.0*B2*A2*A0+B2*A1**2+B1*A3*A0+B1*A2*A1
     1    -2.0*(P0*A3+P1*A2+P2*A1+P3*A0)+A3*A0+A2*A1)/(2.0*A0**2)
      C4=(B4-C3*A1-C2*A2-C1*A3-C0*A4)/A0
      A5=-(5.0*B0*A4*A1+4.0*B1*A4*A0+5.0*B0*A3*A2+4.0*B1*A3*A1
     1    +3.0*B2*A3*A0+2.0*B1*A2**2+3.0*B2*A2*A1+2.0*B3*A2*A0+B3*A1**2
     2    +B4*A1*A0+2.0*(PC*A4+P1*A3+P2*A2+P3*A1+P4*A0+A4*A0+A3*A1)
     3    +A2**2)/(5.0*A0*B0)
      B5=-(8.0*B4*A1*A0+6.0*B3*A2*A0+3.0*B3*A1**2+4.0*B2*A3*A0
     1    +4.C*B2*A2*A1+2.0*B1*A4*A0+2.0*B1*A3*A1+B1*A2**2-4.0*(P0*A4
     2    +P1*A3+P2*A2+P3*A1+P4*AC)+2.0*A4*A0+2.0*A3*A1+A2**2)/
     3    (5.0*A0**2)
      C5=(B5-C4*A1-C3*A2-C2*A3-C1*A4-C0*A5)/A0
C**** EVALUATE POWER SERIES
      XDELT=-4.0*B2*B0/B1**2
      DELT=XDELT*B1/(2.0*B2)/(SQRTF(1.0+XDELT)+1.0)
      T=T+DELT
      PSI=PSIFUN(T)
      ZETA=A0+A1*DELT+A2*DELT**2+A3*DELT**3+A4*DELT**4+A5*DELT**5
      VO=B0+B1*DELT+B2*DELT**2+B3*DELT**3+B4*DELT**4+B5*DELT**5
      UOINC=BC*DELT+B1*DELT**2/2.0+B2*DELT**3/3.0+B3*DELT**4/4.0
     1    +B4*DELT**5/5.0+B5*DELT**6/6.0
      UAINC=C0*DELT+C1*DELT**2/2.0+C2*DELT**3/3.0+C3*DELT**4/4.0
     1    +C4*DELT**5/5.0+C5*DELT**6/6.0
      TS=T/PLASIMP
      VOS=VO/PLASIMP
      GO TO 75
C******** COASTDOWN CALCULATION AFTER PULSE ENDS, T .GT. TF
  110 PRINT 193
      ZETAF=ZETA
      VOF=VO
      UOF=UO
      IF(NZINT.EQ.0) GO TO 109
      DO 111 IZ=1,NZTOT
  111 UF(IZ)=U(IZ)
  109 PSI=0.0
```

```
      AF=(3.0-ZETAF**2)/(VOF*ZETAF)
      BF=ZETAF-AF*TF
      TEND=(SQRTF(3.0)-BF)/AF
      IT=0
  112 IT=IT+1
      T=IT*DT
      IF(T.LE.TF) GO TO 112
  113 ZETA=AF*T+BF
      VO=(3.0-ZETA**2)/(AF*ZETA)
      UO=UOF+((ZETAF**2-ZETA**2)/2.0+3.0*LOGF(ZETA/ZETAF))/AF**2
      IF(ISCALE) GO TO 114
      PRINT 194,T,PSI,UO,VO,ZETA
      GO TO 115
  114 TS=T/PLASIMP
      VOS=VO/PLASIMP
      UOS=UO/PLASIMP**2
      PRINT 194,TS,PSI,UOS,VOS,ZETA
  115 IF(NZINT.EQ.0) GO TO 121
C**** CALCULATE U(Z,T)
      IZ=0
  116 IZ=IZ+1
      Z=IZ*DZ
      IF(Z.GT.ZETAF) GO TO 117
C**** Z IS BETWEEN ZERO AND ZETAF
      U(IZ)=UF(IZ)+((ZETAF-ZETA)*(ZETAF+ZETA-2.0*Z)/2.0
     1  +3.0*LOGF(ZETA/ZETAF)+3.0*Z*((1.0/ZETA)-(1.0/ZETAF)))/AF**2
      GO TO 116
C**** Z IS BETWEEN ZETAF AND ZETA
  117 IF(Z.GT.ZETA) GO TO 118
      U(IZ)=UF(IZ)+(-(ZETA-Z)**2/2.0+3.0*LOGF(ZETA/Z)+3.0*(Z/ZETA-1.0))
     1  /AF**2
      GO TO 116
  118 IF(ISCALE) GO TO 119
      PRINT 190,(U(IZ),IZ=1,NZTOT)
      GO TO 121
  119 DO 120 IZ=1,NZTOT
  120 US(IZ)=U(IZ)/PLASIMP**2
      PRINT 190,(US(IZ),IZ=1,NZTOT)
  121 IF(T.EQ.TEND) GO TO 1
      IT=IT+1
      T=IT*DT
      IF(T.LE.TEND) GO TO 113
      T=TEND
      GO TO 113
C******** MOTION GOES INTO PHASE 2 TYPE DEFORMATION PATTERN
C**** FIND TIME TAU12 AT WHICH MOTION GOES INTO PHASE 2
  122 CONTINUE
      EPSPH2=0.00001
      VO2=VO
      ZETA2=ZETA
      T2=T
      PSI2=PSI
      VO1=FF(1)
      ZETA1=FF(2)
      T1=TT
      PSI1=PSIFUN(T1)
  124 UOP=UO+UCINC
      UOPS=UOP/PLASIMP**2
      IF(ISCALE) GO TO 125
      PRINT 188,T,PSI,UOP,VO,ZETA,NOFNS
      GO TO 126
  125 PRINT 188,TS,PSI,UOPS,VOS,ZETA,NOFNS
```

```
  126 H=(T2-T1)*(1.5-PSI1*ZETA1)/(PSI2*ZETA2-PSI1*ZETA1)
      DO 127 I=1,4
  127 F(I)=FF(I)
      T=T1
      CALL DIFSUB(NEQ,T,F,DF,H,DTMIN,EPSERR,MORD,METH,FMAX,RLTVER,KKSTP)
      PSI=PSIFUN(T)
      IF(KKSTP.LT.0) GO TO 72
      VO=F(1)
      ZETA=F(2)
      UOINC=F(3)
      UAINC=F(4)
      TS=T/PLASIMP
      VOS=VO/PLASIMP
      IF(ABSF(PSI*ZETA-1.5).LT.EPSPH2) GO TO 130
      IF(PSI*ZETA.GT.1.5) GO TO 129
      T1=T
      ZETA1=ZETA
      PSI1=PSI
      DO 128 I=1,4
  128 FF(I)=F(I)
      GO TO 124
  129 T2=T
      ZETA2=ZETA
      PSI2=PSI
      GO TO 124
  130 CONTINUE
      UO=UO+UOINC
      UOS=UO/PLASIMP**2
      DO 131 IZ=1,NZ
      U(IZ)=U(IZ)+UOINC-IZ*DZ*UAINC
      US(IZ)=U(IZ)/PLASIMP**2
  131 CONTINUE
      TAU12=T
      PRINT 195,T,TS
      ETA=ZETA
      IF(ISCALE) GO TO 132
      PRINT 196,T,PSI,UO,VO,ZETA,ETA,NOFNS
      PRINT 190,(U(IZ),IZ=1,NZTOT)
      GO TO 133
  132 PRINT 196,TS,PSI,UOS,VOS,ZETA,ETA,NOFNS
      PRINT 190,(US(IZ),IZ=1,NZTOT)
  133 CONTINUE
      KPHASE=1
      K21=0
      K22=0
      F(1)=VO
      F(2)=VAUX=VO/ZETA
      JOM=1001
      OMEGAL(JOM)=TAU12
      ZL(JOM)=ZETA
      GO TO 229
C**** AN INSTANTANEOUS JUMP INTO PHASE 2 OCCURS AT T
  222 TAU12=T
      PRINT 199,T,TS
      ZETA1=ZETA2=ETA2=ETA=ZETA
      PSI2=1.5/ZETA2
      K21=K22=0
      PSI=PSI+PJMP
      ZETA=1.5/PSI
      IF(ISCALE) GO TO 223
      PRINT 196,T,PSI2,UO,VO,ZETA2,ETA2,NOFNS
      PRINT 190,(U(IZ),IZ=1,NZTOT)
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      PRINT 196,T,PSI,UO,VO,ZETA,ETA,NOFNS
      PRINT 190,(U(IZ),IZ=1,NZTOT)
      GO TO 224
  223 PRINT 196,TS,PSI2,UOS,VOS,ZETA2,ETA2,NOFNS
      PRINT 190,(US(IZ),IZ=1,NZTOT)
      PRINT 196,TS,PSI,UOS,VOS,ZETA,ETA,NOFNS
      PRINT 190,(US(IZ),IZ=1,NZTOT)
  224 VAUX2=VAUX=VO/ZETA2
      VZETA2=0
      VZETA=VO-VAUX*ZETA
      OMEGAL(1001)=TAU12
      ZL(1001)=ZETA2
      OMEGAL(1000)=VZETA+TAU12
      ZL(1000)=ZETA
      JOM=1000
      F(1)=VO
      F(2)=VAUX
      JJMP=JJMP+1
      IF(TAU12.EQ.TDT) IT=IT+1
      IF(TAU12.EQ.TAUMAX) GO TO 226
      KPHASE=1
      GO TO 229
  226 JMAX=JMAX+1
      KPHASE=2
  229 CONTINUE
C******** PHASE 2 DEFORMATION
C**** INTEGRATION TIME STEPS AND PRINTOUT TIMES
      NT2=10
      NT3=2
      DT2=DT/NT2
      DT3= DT2/NT3
      TDT=IT*DT
      DO 231 IT2=1,NT2
      TDT2=TDT-(NT2-IT2)*DT2
      IF(TDT2.GT.TAU12) GO TO 232
  231 CONTINUE
      GO TO 1
  232 DO 233 IT3=1,NT3
      TDT3=TDT2-(NT3-IT3)*DT3
      IF(TDT3.GT.TAU12) GO TO 235
  233 CONTINUE
      GO TO 1
  235 IF(JJMP.GT.NOJMP) 236,237
  236 TAUJMP=2.0*PLASIMP
      PJMP=0.0
      GO TO 238
  237 TAUJMP=TJMP(JJMP)
      PJMP=PSIJMP(JJMP)
  238 IF(JMAX.GT.NOMAX) 239,240
  239 TAUMAX=2.0*PLASIMP
      PMAX=0.0
      GO TO 241
  240 TAUMAX=TMAX(JMAX)
      PMAX=PSIMAX(JMAX)
  241 TDT=IT*DT
  242 TDT2=TDT-(NT2-IT2)*DT2
  243 TDT3=TDT2-(NT3-IT3)*DT3
      TPR=AMIN1(TDT2,TAUJMP,TAUMAX)
  244 TCAL=AMIN1(TDT3,TAUJMP,TAUMAX)
C**** USE SUBROUTINES DIFSUB AND DIFFUN
  246 T1=T
      PSI1=PSI
```

```
      ZETA1=ZETA
      ETA1=ETA
      VO1=VO
      VAUX1=VAUX
  249 F(3)=F(4)=0.0
  250 H=TCAL-T
      CALL DIFSUB(NEQ,T,F,DF,H,DTMIN,EPSERR,MORD,METH,FMAX,RLTVER,KKSTP)
      PSI=PSIFUN(T)
      IF(T.EQ.TAUJMP) PSI=PSI-PJMP
      IF(KKSTP.LT.0) GO TO 72
      IF(ABSF(TCAL-T).GT.EPSPR) GO TO 250
      VO=F(1)
      VAUX=F(2)
      UOINC=F(3)
      UAINC=F(4)
      TS=T/PLASIMP
      VOS=VO/PLASIMP
      IF(KPHASE.EQ.2) GO TO 253
C**** T IS BETWEEN TAU12 AND TAUMAX (KPHASE=1)
  252 ZETA=1.5/PSI
      VZETA=VC-VAUX*ZETA
      JOM=JOM-1
      OMEGAL(JOM)=VZETA+T
      ZL(JOM)=ZETA
      ETA=OMEGINV(T)
      GO TO 265
C**** T IS BETWEEN TAUMAX AND TAU21 (KPHASE=2)
  253 ZETA=ZETAEQ(VAUX,VO+T)
      ETA=OMEGINV(T)
      VZETA=VC-VAUX*ZETA
      IF(K21) GO TO 255
      IF(K22) GO TO 258
      IF(ETA.LE.ZETA) GO TO 254
      IF(PSI*ZETA.GT.1.50001) GO TO 257
      GO TO 265
C**** T IS IN THE VICINITY OF TAU21
  254 K21=1
  255 IF(ABSF(ETA-ZETA).LE.EPSZ) GO TO 259
      TCAL=(T*(ETA1-ZETA1)+T1*(ZETA-ETA))/(ETA1-ZETA1+ZETA-ETA)
      T=T1
      F(1)=VO1
      F(2)=VAUX1
      GO TO 249
C**** INTERPOLATE TO PSI*ZETA=1.5
  257 K22=1
  258 IF(ABSF(PSI*ZETA-1.5).LE.EPSZ) GO TO 259
      TCAL=((PSI*ZETA-1.5)*ZETA1*T1+(1.5-PSI1*ZETA1)*ZETA*T)/
     1    ((PSI*ZETA-1.5)*ZETA1+(1.5-PSI1*ZETA1)*ZETA)
      T=T1
      F(1)=VO1
      F(2)=VAUX1
      GO TO 249
  259 TPR=T
      GO TO 265
C**** DEFORMATION CALCULATIONS
  265 UO=UO+UOINC
      UOS=UO/PLASIMP**2
      DO 268 IZ=1,NZTOT
      Z=IZ*DZ
      IF(Z.GT.ZETA) GO TO 266
      U(IZ)=U(IZ)+UOINC-Z*UAINC
      GO TO 267
```

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  266 IF(Z.GT.ETA) GO TO 268
      U(IZ)=U(IZ)+(-0.5*(T+T1)+OMEGA(Z))*(T-T1)
  267 US(IZ)=U(IZ)/PLASIMP**2
  268 CONTINUE
C**** PRINTOUTS
      IF(ABSF(T-TPR).LE.EPSPR) T=TPR
      IF(T.EQ.TPR) 269,272
  269 IF(ISCALE) GO TO 270
      PRINT 196,T,PSI,UO,VO,ZETA,ETA,NOFNS
      PRINT 190,(U(IZ),IZ=1,NZTOT)
      GO TO 275
  270 PRINT 196,TS,PSI,UOS,VOS,ZETA,ETA,NOFNS
      PRINT 190,(US(IZ),IZ=1,NZTOT)
      GO TO 275
C**** BRANCHING
  272 IT3=IT3+1
      GO TO 243
  275 IF(K21) GO TO 277
      IF(K22) GO TO 280
      IF(T.EQ.TAUJMP) GO TO 285
      IF(T.EQ.TAUMAX) GO TO 295
      IF(T.EQ.TOT2) GO TO 297
      GO TO 1
C**** T EQUALS TAU21 (RETURN TO PHASE 1 CALCULATION)
  277 KPHASE=0
      PRINT 197,T,TS
      IF(T.GE.TF) GO TO 279
      F(1)=VO
      F(2)=ZETA
      NZ=NZZ=NZTOT
  278 IF(NZ*DZ.LE.ZETA) GO TO 70
      NZ=NZZ=NZ-1
      GO TO 278
  279 TF=T
      GO TO 110
C**** T EQUALS A NEW TAU12 (HINGE BAND BEGINS TO EXPAND AGAIN)
  280 TAU12=T
      KPHASE=1
      K22=0
      PRINT 198,T,TS
      DO 281 JZ=JOM,1000
      IF(ZETA.GE.ZL(JZ).AND.ZETA.LE.ZL(JZ+1))GO TO 282
  281 CONTINUE
      JOM=JOM-1
      OMEGAL(JOM)=VZETA+T
      ZL(JOM)=ZETA
      GO TO 242
  282 OM=OMEGA(ZETA)
      JOM=JZ-1
      OMEGAL(JZ)=OM
      OMEGAL(JOM)=VZETA+T
      ZL(JZ)=ZL(JOM)=ZETA
      GO TO 242
C**** T EQUALS TAUJMP
  285 JJMP=JJMP+1
      PSI1=PSI
      PSI=PSI+PJMP
      IF(PJMP.LE.0.0) GO TO 292
      IF(KPHASE.EQ.2) GO TO 286
      JOM=JOM-1
      ZETA=1.5/PSI
      VZETA=VO-VAUX*ZETA
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      OMEGAL(JOM)=VZETA+T
      ZL(JOM)=ZETA
      GO TO 292
  286 IF(PSI*ZETA.LT.1.5) GO TO 292
      PSI2=1.5/ZETA
      ZETA2=ZETA
      VZETA2=VZETA=VO-VAUX*ZETA
      PRINT 198,T,TS
      IF(ISCALE) GO TO 287
      PRINT 196,T,PSI2,UO,VO,ZETA2,ETA,NOFNS
      PRINT 190,(U(IZ),IZ=1,NZTOT)
      GO TO 288
  287 PRINT 196,TS,PSI2,UOS,VOS,ZETA2,ETA,NOFNS
      PRINT 190,(US(IZ),IZ=1,NZTOT)
  288 TAU12=T
      DO 289 JZ=JOM,1000
      IF(ZETA2.GE.ZL(JZ).AND.ZETA2.LE.ZL(JZ+1)) GO TO 290
  289 CONTINUE
      JOM=JOM-1
      OMEGAL(JOM)=VZETA2+T
      ZL(JOM)=ZETA2
      GO TO 291
  290 OM=OMEGA(ZETA2)
      JOM=JZ-1
      OMEGAL(JZ)=OM
      OMEGAL(JOM)=VZETA2+T
      ZL(JZ)=ZL(JOM)=ZETA2
  291 ZETA=1.5/PSI
      VZETA=VO-VAUX*ZETA
      KPHASE=1
      JOM=JOM-1
      OMEGAL(JOM)=VZETA+T
      ZL(JOM)=ZETA
  292 IF(ISCALE) GO TO 293
      PRINT 196,T,PSI,UO,VO,ZETA,ETA,NOFNS
      PRINT 190,(U(IZ),IZ=1,NZTOT)
      GO TO 294
  293 PRINT 196,TS,PSI,UOS,VOS,ZETA,ETA,NOFNS
      PRINT 190,(US(IZ),IZ=1,NZTOT)
  294 IF(T.NE.TAUMAX) GO TO 296
  295 JMAX=JMAX+1
      KPHASE=2
  296 IF(T.EQ.TOT2) 297,235
  297 IT3=1
      IF(IT2.EQ.NT2) GO TO 298
      IT2=IT2+1
      GO TO 235
  298 IT2=1
      IT=IT+1
      GO TO 235
C******** FORMATS
  170 FORMAT(10A8)
  171 FORMAT(1H1,7X*PROGRAM RINGLOAD*//8X,10A8)
  172 FORMAT(I2,2X,I2,2X,I2)
  173 FORMAT(1H0,7X*INPUT ERROR*/8X*KSTOP =*I2)
  174 FORMAT(1H0,7X*INSTJMP =*I2
     1    *, INDICATING AN INSTANTANEOUS JUMP AT TYIELD*)
  175 FORMAT(1H0,7X*PULSE PARAMETERS*/12X
     1    *PULSE BEGINS AT TAUO =*E12.5/12X
     2    *PULSE ENDS AT TAUF =*E12.5/12X
     3    *PLASTIC FLOW STARTS AT TY =*E12.5/12X
     4    *TOTAL IMPULSE TOTIMP =*E12.5/12X
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    5     *PLASTIC IMPULSE (IMPULSE AFTER TY) PLASIMP = I =*E12.5/12X
    6     *MEAN TIME OF PULSE TM =*E12.5/12X
    7     *EFFECTIVE LOAD PSIE =*E12.5)
  176 FORMAT(1H0,7X*SCALED PULSE PARAMETERS*/12X
    1     *TAUC/I =*E12.5/12X*TAUF/I =*E12.5/12X
    2     *TY/I =*E12.5/12X*TM/I =*E12.5)
  177 FORMAT(1H0,7X*LOCATIONS OF RELATIVE MAXIMA OF PULSE*/3X*K*7X
    1     *TMAX*12X*PSIMAX*11X*TMAX/I*/(I4,3E17.7))
  178 FORMAT(1H0,7X*THE SELECTED OUTPUT INTERVALS ARE*/12X
    1     *TIME INTERVAL DT =*E12.5/12X
    2     *SCALED TIME INTERVAL DTS = DT/I =*E12.5/12X
    3     *AXIAL INTERVAL DZ =*E12.5)
  179 FORMAT(1H0,7X*SELECTED SCALED TIME INTERVAL IS TOO LARGE,*
    1     /8X*I.E., NTSINT IS SMALLER THAN TEN.*)
  180 FORMAT(1H0,7X*INSTANTANEOUS JUMPS IN PULSE SHAPE*/3X*K*7X
    1     *TJMP*12X*PSIJMP*11X*TJMP/I*/(I4,3E17.7))
  181 FORMAT(1H0,7X*SELECTED AXIAL INTERVAL FOR DISPLACEMENT PRINTOUT *
    1     *IS TOO SMALL,*/8X*I.E., NZINT IS LARGER THAN 25*)
  182 FORMAT(1H1,8X*T/I*12X*PSI(T)*9X*U(0,T)/I/I*8X*V(0,T)/I*10X
    1     *ZETA(T)*10X*ETA(T)*11X*NOFNS*)
  183 FORMAT(1H0,37X*U(1DZ,T)/I/I*5X*U(2DZ,T)/I/I*5X*U(3DZ,T)/I/I*5X
    1     *U(4DZ,T)/I/I*5X*U(5DZ,T)/I/I*/38X*U(6DZ,T)/I/I*5X
    2     *U(7DZ,T)/I/I*5X*U(8DZ,T)/I/I*5X*U(9DZ,T)/I/I*5X
    3     *U(10DZ,T)/I/I*)
  184 FORMAT(1H1,9X*T*13X*PSI(T)*11X*U(0,T)*11X*V(0,T)*11X*ZETA(T)*10X
    1     *ETA(T)*11X*NOFNS*)
  185 FORMAT(1H0,39X*U(1DZ,T)*9X*U(2DZ,T)*9X*U(3DZ,T)*9X*U(4DZ,T)*9X
    1     *U(5DZ,T)*/40X*U(6DZ,T)*9X*U(7DZ,T)*9X*U(8DZ,T)*9X*U(9DZ,T)*
    2     9X*U(10DZ,T)*)
  187 FORMAT(1H0,/5E17.7,17X,I10)
  188 FORMAT(1H0,/2X,5E17.7,17X,I10)
  189 FORMAT(1H0,35X,5E17.7/(36X,5E17.7))
  190 FORMAT(1H0,33X,5E17.7/(34X,5E17.7))
  191 FORMAT(1H0,7X*NO CONVERGENCE IN STEP TO T =*E12.5)
  192 FORMAT(1H0,7X*DEFORMATION STOPS BEFORE END OF PULSE*)
  193 FORMAT(1H0,7X*DEFORMATION CONTINUES AFTER END OF PULSE*)
  194 FORMAT(1H0,/5E17.7)
  195 FORMAT(1H0,//8X*A HINGE BAND FORMS AT T =*E12.5*, T/I =*E12.5)
  196 FORMAT(1H0,/6E17.7,I10)
  197 FORMAT(1H0,//8X*THE HINGE BAND REDUCES TO A HINGE CIRCLE AT T =*
    1     E12.5*, T/I =*E12.5)
  198 FORMAT(1H0,//8X*THE HINGE BAND BEGINS TO EXPAND AGAIN AT T =*
    1     E12.5*, T/I =*E12.5)
  199 FORMAT(1H0,//8X*A HINGE BAND FORMS INSTANTANEOUSLY AT T =*
    1     E12.5*, T/I =*E12.5)
      END
```

```
      SUBROUTINE DIFFUN(T,F,DF)
C
C     SUBROUTINE WHICH PROVIDES RINGLOAD PROGRAM WITH RIGHTHAND SIDES OF
C         DIFFERENTIAL EQUATIONS
C
      DIMENSION F(4),DF(4),SPSI(4)
      COMMON/COMDIF/NOFNS,KPHASE,TAUJMP,PJMP
      PSI=PSIFUN(T)
      IF(T.EQ.TAUJMP) PSI=PSI-PJMP
C**** F(1)=VO, F(3)=UOINC, F(4)=UAINC
      IF(KPHASE-1)400,401,402
C**** MOTION IS IN PHASE 1 (KPHASE=0)
  400 VO=F(1)
      ZETA=F(2)
      DF(1)=(4.0*PSI-3.0/ZETA-ZETA)/ZETA
      DF(2)=(-2.0*PSI+3.0/ZETA-ZETA)/VO
      DF(3)=VO
      DF(4)=VO/ZETA
      GO TO 404
C**** MOTION IS IN PHASE 2 (KPHASE=1)
  401 VO=F(1)
      VAUX=F(2)
      ZETA=1.5/PSI
      GO TO 403
C**** MOTION IS IN PHASE 2 (KPHASE=2)
  402 VO=F(1)
      VAUX=F(2)
      ZETA=ZETAEQ(VAUX,VO+T)
  403 DF(1)=(4.0*PSI-3.0/ZETA-ZETA)/ZETA
      DF(2)=6.0*(PSI*ZETA-1.0)/ZETA**3
      DF(3)=VO
      DF(4)=VAUX
  404 NOFNS=NOFNS+1
      END
```

```
      FUNCTION OMEGA(Z)
C
C     FINDS OMEGA(Z) BY LINEAR INTERPOLATION IN THE TABLE OMEGAL
C
      COMMON/COMOMEG/JOM,OMEGAL(1001),ZL(1001)
      DO 500 JZ=JOM,1000
      IF(ZL(JZ).EQ.ZL(JZ+1)) GO TO 500
      IF(Z.GE.ZL(JZ).AND.Z.LE.ZL(JZ+1)) GO TO 501
  500 CONTINUE
      IF(Z.GT.ZL(1001)) 506,507
  506 PRINT 510,Z,(JZ,ZL(JZ),OMEGAL(JZ),JZ=JOM,1001)
      STOP
  507 CONTINUE
      IF(Z.LT.ZL(JOM)) JZ=JOM
  501 OMEGA=(OMEGAL(JZ+1)*(Z-ZL(JZ))+OMEGAL(JZ)*(ZL(JZ+1)-Z))
     1     /(ZL(JZ+1)-ZL(JZ))
  510 FORMAT(1H0,7X*OMEGA STOP, Z =*E12.5/(8X,I6,2E17.7))
      END
```

```
      FUNCTION OMEGINV(X)
C
C     SOLVES X = OMEGA(Z) FOR Z
C
      COMMON/CCMOMEG/JOM,OMEGAL(1001),ZL(1001)
      DO 550 JZ=JOM,1000
      IF(OMEGAL(JZ).EQ.OMEGAL(JZ+1)) GO TO 550
      IF(X.LE.OMEGAL(JZ).AND.X.GE.OMEGAL(JZ+1)) GO TO 551
  550 CONTINUE
      IF(X.LT.OMEGAL(1001)) 556,557
  556 PRINT 560,X,(JZ,ZL(JZ),OMEGAL(JZ),JZ=JOM,1001)
      STOP
  557 CONTINUE
      IF(X.GT.OMEGAL(JOM)) JZ=JOM
  551 OMEGINV=(ZL(JZ+1)*(X-OMEGAL(JZ))+ZL(JZ)*(OMEGAL(JZ+1)-X))
     1     /(OMEGAL(JZ+1)-OMEGAL(JZ))
  560 FORMAT(1H0,7X*OMEGINV STOP, T =*E12.5/(8X,I6,2E17.7))
      END
```

```
      FUNCTION ZETAEQ(A,B)
C
C     SOLVES THE EQUATION OMEGA(Z) = B - A*Z FOR Z, WHERE
C          OMEGA(Z) = AOM(J)*Z + BOM(J)
C
      COMMON/CCMOMEG/JOM,OMEGAL(1001),ZL(1001)
      DO 600 JZ=JOM,1000
      IF(ZL(JZ).EQ.ZL(JZ+1)) GO TO 600
      AOM=(OMEGAL(JZ+1)-OMEGAL(JZ))/(ZL(JZ+1)-ZL(JZ))
      BOM=(OMEGAL(JZ)*ZL(JZ+1)-OMEGAL(JZ+1)*ZL(JZ))/(ZL(JZ+1)-ZL(JZ))
      IF(ABSF(A+AOM).GT.0.000001) GO TO 603
      ZETAEQ=ZL(JOM)
      RETURN
  603 CONTINUE
      Z=(B-BOM)/(A+AOM)
      IF(Z.GE.ZL(JZ).AND.Z.LE.ZL(JZ+1)) GO TO 601
  600 CONTINUE
      AOM=(OMEGAL(JOM+1)-OMEGAL(JOM))/(ZL(JOM+1)-ZL(JOM))
      BOM=(OMEGAL(JOM)*ZL(JOM+1)-OMEGAL(JOM+1)*ZL(JOM))
     1     /(ZL(JOM+1)-ZL(JOM))
      Z=(B-BOM)/(A+AOM)
      IF(Z.GT.ZL(JOM)) 606,601
  606 PRINT 610,Z,(JZ,ZL(JZ),OMEGAL(JZ),JZ=JOM,1001)
      STOP
  601 ZETAEQ=Z
  610 FORMAT(1H0,7X*ZETAEQ STOP, Z =*E12.5/(8X,I6,2E17.7))
      END
```

```
      SUBROUTINE PULSEINF(TO,TF,TYIELD,TOTIMP,PLASIMP,TMEAN,NOMAX,
     1     TMAX,PSIMAX,INSTJMP,NOJMP,TJMP,PSIJMP,KSTOP)
C
C     SUBROUTINE WHICH PROVIDES RINGLOAD PROGRAM WITH BASIC INFORMATION
C         ON PULSE SHAPE.
C
C     PIECEWISE LINEAR VERSION.
C     THE PULSE SHAPE IS DESCRIBED BY GIVING PSI VALUES AT A NUMBER OF
C         TIME POINTS AND USING LINEAR INTERPOLATION.
C
C     REQUIRED INPUT FROM DATA CARDS
C     CARD 3.   FORMAT(I2)
C         NJ (NUMBER OF INPUT DATA POINTS)
C     CARDS 4,5,6....   FORMAT(2E15.7)
C         TJ(J) AND PSIJ(J) (DATA POINT PAIRS)
C
C     OUTPUT TO RINGLOAD
C         TO (TIME AT WHICH PULSE BEGINS)
C         TF (TIME AT WHICH PULSE ENDS)
C         TYIELD (TIME AT WHICH PLASTIC FLOW FIRST OCCURS, PSI=1.0)
C         TOTIMP (TOTAL IMPULSE)
C         PLASIMP (IMPULSE AFTER TYIELD)
C         TMEAN (TIME MEAN OF PLASTIC IMPULSE MEASURED FROM TYIELD)
C         NOMAX (NUMBER OF RELATIVE MAXIMA)
C         TMAX(K) AND PSIMAX(K) (LOCATIONS AND VALUES OF RELATIVE MAXIMA)
C         INSTJMP (IF POSITIVE,PULSE HAS INSTANTANEOUS JUMP AT TYIELD)
C         NOJMP (NUMBER OF TIMES AT WHICH PSI HAS AN INSTANTANEOUS JUMP)
C         TJMP(K) AND PSIJMP(K) (LOCATIONS AND VALUES OF JUMPS)
C         KSTOP (IF POSITIVE, THERE IS AN INPUT ERROR)
C
C     COMMON WITH FUNCTION PSIFUN AND SUBROUTINE PSICOEF
C         NJ, TJ(J), PSIJ(J)
C
C     IF AN INSTANTANEOUS JUMP OCCURS AT TJ, THE VALUE OF PSI TO
C         THE LEFT OF THE JUMP IS RETURNED.
C
      DIMENSION TJS(99),TJMP(50),PSIJMP(50),TMAX(50),PSIMAX(50)
      COMMON/PULSE/NJ,TJ(99),PSIJ(99)
C******** TITLE, DATA INPUT, AND INITIALIZATIONS
      READ 286,NJ
      READ 285,(TJ(J),PSIJ(J),J=1,NJ)
      PRINT 290
      KSTOP=0
      INSTJMP=0
C******** CHECK ORDERING OF INPUT CARDS
      DO 201 J=2,NJ
      IF(TJ(J).LT.TJ(J-1)) GO TO 280
  201 CONTINUE
C******** TO AND TF
      TO=TJ(1)
      TF=TJ(NJ)
C******** TYIELD AND INSTJMP
      IF(PSIJ(1).GT.1.0) GO TO 204
      DO 203 J=2,NJ
      IF(PSIJ(J).GT.1.0.AND.PSIJ(J-1).LE.1.0) GO TO 205
  203 CONTINUE
      GO TO 281
  204 INSTJMP=1
      JYIELD=1
      TYIELD=TO
      GO TO 209
  205 JYIELD=J
```

```
      IF(TJ(J).EQ.TJ(J-1)) GO TO 206
      TYIELD=(TJ(J-1)*(PSIJ(J)-1.0)+TJ(J)*(1.0-PSIJ(J-1)))/(PSIJ(J)-
     1    PSIJ(J-1))
      GO TO 209
  206 INSTJMP=1
      TYIELD=TJ(J)
  209 CONTINUE
C******** TOTIMP,PLASIMP,TMEAN
      TOTIMP=0.0
      DO 210 J=2,NJ
  210 TOTIMP=TOTIMP+(PSIJ(J)+PSIJ(J-1))*(TJ(J)-TJ(J-1))/2.0
      IF(INSTJMP) GO TO 212
      PLASIMP=(PSIJ(JYIELD)+1.0)*(TJ(JYIELD)-TYIELD)/2.0
      AUXTM=(2.0*PSIJ(JYIELD)+1.0)*(TJ(JYIELD)-TYIELD)**2/6.0
      GO TO 213
  212 PLASIMP=0.0
      AUXTM=0.0
  213 JJ=JYIELD+1
      DO 214 J=JJ,NJ
      PLASIMP=PLASIMP+(PSIJ(J)+PSIJ(J-1))*(TJ(J)-TJ(J-1))/2.0
      AUXTM=AUXTM+(PSIJ(J)*(2.0*TJ(J)+TJ(J-1)-3.0*TYIELD)+PSIJ(J-1)*
     1    (TJ(J)+2.0*TJ(J-1)-3.0*TYIELD))*(TJ(J)-TJ(J-1))/6.0
  214 CONTINUE
      TMEAN=AUXTM/PLASIMP
      EFFWIDTH=2.0*TMEAN
C******** NOMAX, TMAX(K), AND PSIMAX(K)
      NOMAX=0
      IF(PSIJ(2).GT.PSIJ(1)) GO TO 221
      NOMAX=1
      PSIMAX(1)=PSIJ(1)
      TMAX(1)=TJ(1)
  221 DO 223 J=3,NJ
      IF(PSIJ(J-1).GT.PSIJ(J-2).AND.PSIJ(J).LE.PSIJ(J-1))222,223
  222 NOMAX=NOMAX+1
      PSIMAX(NOMAX)=PSIJ(J-1)
      TMAX(NOMAX)=TJ(J-1)
  223 CONTINUE
      IF(PSIJ(NJ).GT.PSIJ(NJ-1))224,225
  224 NOMAX=NOMAX+1
      PSIMAX(NOMAX)=PSIJ(NJ)
      TMAX(NOMAX)=TJ(NJ)
  225 CONTINUE
C******** SCALING OF TIMES BY PLASTIC IMPULSE
      DO 230 J=1,NJ
  230 TJS(J)=TJ(J)/PLASIMP
C******** NOJMP, TJMP(K), AND PSIJMP(K)
      NOJMP=0
      DO 241 J=2,NJ
      IF(TJ(J).EQ.TJ(J-1)) 240,241
  240 NOJMP=NOJMP+1
      TJMP(NOJMP)=TJ(J)
      PSIJMP(NOJMP)=PSIJ(J)-PSIJ(J-1)
  241 CONTINUE
C******** PRINTOUTS OF PULSE DATA AND ERROR MESSAGES
      PRINT 294,(J,TJ(J),PSIJ(J),TJS(J),J=1,NJ)
      RETURN
  280 KSTOP=1
      PRINT 291
      PRINT 292,(J,TJ(J),PSIJ(J),J=1,NJ)
      RETURN
  281 KSTOP=1
      PRINT 293
```

```
      PRINT 292,(J,TJ(J),PSIJ(J),J=1,NJ)
      RETURN
C******** FORMATS
  285 FORMAT(2E15.7)
  286 FORMAT(I2)
  290 FORMAT(1H0,7X*SUBROUTINE PULSEINF IS PIECEWISE LINEAR VERSION*/8X
     1     *THE PULSE SHAPE IS DESCRIBED BY GIVING VALUES OF PSI(J)*/8X
     2     *AT NJ POINTS OF TIME T(J) AND USING LINEAR INTERPOLATION*)
  291 FORMAT(1H0,7X*INPUT CARDS ARE OUT OF ORDER*)
  292 FORMAT(1H0,7X*PULSE INPUT DATA*/3X*J*8X*T*15X*PSI*/(I4,2E17.7))
  293 FORMAT(1H0,7X*NO PLASTIC FLOW OCCURS (PSI NEVER EXCEEDS 1.0)*)
  294 FORMAT(1H0,7X*PULSE INPUT DATA*/3X*J*8X*T*15X*PSI*14X*T/I*/
     1     (I4,3E17.7))
      END
```

```
      FUNCTION PSIFUN(T)
C
C     SUBROUTINE WHICH PROVIDES RINGLOAD PROGRAM WITH VALUES OF PSI.
C
C     PIECEWISE LINEAR VERSION.
C     THE PULSE SHAPE IS DESCRIBED BY GIVING PSI VALUES AT A NUMBER OF
C         TIME POINTS AND USING LINEAR INTERPOLATION.
C
C     INPUT
C         T (TIME AT WHICH PSI IS WANTED)
C
C     OUTPUT
C         PSI (PULSE MAGNITUDE AT T)
C
C     COMMON WITH SUBROUTINE PULSEINF
C         NJ, TJ(J), PSIJ(J)
C
      COMMON/PULSE/NJ,TJ(99),PSIJ(99)
      PSIFUN=0.0
      IF(T.LT.TJ(1).OR.T.GE.TJ(NJ)) RETURN
      DO 304 J=2,NJ
      IF(T.GE.TJ(J-1).AND.T.LT.TJ(J)) GO TO 305
  304 CONTINUE
  305 PSIFUN=(PSIJ(J-1)*(TJ(J)-T)+PSIJ(J)*(T-TJ(J-1)))/(TJ(J)-TJ(J-1))
      END
```

```
      SUBROUTINE PSICOEF(T,SPSI)
C
C     SUBROUTINE WHICH PROVIDES RINGLOAD PROGRAM WITH THE COEFFICIENTS
C         OF THE POWER SERIES EXPANSION OF PULSE SHAPE PSI.
C
C     PIECEWISE LINEAR VERSION.
C     THE PULSE SHAPE IS DESCRIBED BY GIVING PSI VALUES AT A NUMBER OF
C         TIME POINTS AND USING LINEAR INTERPOLATION.
C
C     INPUT
C         T (TIME)
C
C     OUTPUT
C         SPSI(N) (NTH DERIVATIVE CF PSI DIVIDED BY N FACTORIAL AND
C             EVALUATED AT T FOR N = 1,2,3,4)
C
C     COMMON WITH SUBROUTINE PULSEINF
C         NJ, TJ(J), PSIJ(J)
C
      DIMENSICN SPSI(4)
      COMMON/PULSE/NJ,TJ(99),PSIJ(99)
      SPSI(1)=SPSI(2)=SPSI(3)=SPSI(4)=0.0
      IF(T.LT.TJ(1).OR.T.GE.TJ(NJ)) RETURN
      DO 354 J=2,NJ
      IF(T.GE.TJ(J-1).AND.T.LT.TJ(J)) GO TO 355
  354 CONTINUE
  355 SPSI(1)=(PSIJ(J)-PSIJ(J-1))/(TJ(J)-TJ(J-1))
      END



      SUBROUTINE PULSEINF(TAUO,TAUF,TYIELD,TOTIMP,PLASIMP,TMEAN,NOMAX,
     1    TMAX,PSIMAX,INSTJMP,NOJMP,TJMP,PSIJMP,KSTOP)
C
C     SUBROUTINE WHICH PROVIDES RINGLOAD PROGRAM WITH BASIC INFORMATION
C         ON PULSE SHAPE.
C
C     EXPONENTIAL DECAY VERSION.
C     THE PULSE IS A LINEAR RISE FCLLOWED BY AN EXPONENTIAL DECAY
C         PSI=PSIM*(T-TAUO)/(TAUM-TAUO), TAUO.LE.T.LE.TAUM
C         PSI=PSIM*EXPF(-(T-TAUM)/TAU), TAUM.LT.T.LE.TAUF
C
C     REQUIREC INPUT FROM DATA CARC AND COMMON WITH PSIFUN AND PSICOEF
C         TAUC,TAUM,TAUF,TAU,PSIM
C
      DIMENSION TMAX(1),PSIMAX(1),TJMP(1),PSIJMP(1)
      COMMON/PULSE/TO,TM,TF,TAU,PSIM
C******** DATA INPUT AND TITLE
      READ 290,TO,TM,TF,TAU,PSIM
      PRINT 291,TO,TM,TF,TAU,PSIM
      TAUO=TO
      TAUF=TF
      KSTOP=0
      IF(PSIM.GT.1.0) GO TO 203
      KSTOP=1
      PRINT 292
      RETURN
C******** TYIELD, INSTJMP,NOMAX,TMAX,PSIMAX,NOJMP,TJMP,PSIJMP
  203 IF(TM.GE.TO.AND.TF.GE.TM) GO TO 205
```

```
      KSTOP=1
      PRINT 293
      RETURN
  205 NOMAX=1
      PSIMAX(1)=PSIM
      TMAX(1)=TM
      IF(TM.GT.TO) GO TO 208
      INSTJMP=NOJMP=1
      TJMP(1)=TYIELD=TO
      PSIJMP(1)=PSIM
      GO TO 210
  208 INSTJMP=NOJMP=0
      TYIELD=TO+(TM-TO)/PSIM
C******** TOTIMP AND PLASIMP
  210 EXPINT=PSIM*TAU*(1.0-EXPF((TM-TF)/TAU))
      IF(TM.GT.TO) GO TO 212
      TOTIMP=PLASIMP=EXPINT
      GO TO 215
  212 TOTIMP=PSIM*(TM-TO)/2.0+EXPINT
      PLASIMP=PSIM*(TM-TYIELD)*(TM+TYIELD-2.0*TO)/(2.0*(TM-TO))+EXPINT
C******** TMEAN
  215 TEXPINT=PSIM*TAU*(TM+TAU-TYIELD-(TF+TAU-TYIELD)*EXPF((TM-TF)/TAU))
      IF(TM.GT.TO) GO TO 217
      TMEAN=TEXPINT/PLASIMP
      GO TO 219
  217 TTINT=PSIM*(2.0*TM**3-3.0*(TYIELD+TO)*TM**2+6.0*TYIELD*TO*TM
     1    +(TYIELD-3.0*TO)*TYIELD**2)/(6.0*(TM-TO))
      TMEAN=(TTINT+TEXPINT)/PLASIMP
  219 CONTINUE
      RETURN
C******** FORMATS
  290 FORMAT(5E15.7)
  291 FORMAT(1H0,7X*SUBROUTINE PULSEINF IS EXPONENTIAL DECAY VERSION*/8X
     1    *THE PULSE IS A LINEAR RISE FOLLOWED BY AN EXPONENTIAL DECAY*
     2    /13X*PSI = (PSIM)(T-TAUO)/(TAUM-TAU), TAUO.LE.T.LE.TAUM,*
     3    /13X*PSI = (PSIM)EXPF(-(T-TAUM)/TAU), TAUM.LT.T.LE.TAUF,*
     4    /13X*TAUO =*E12.5*, TAUM =*E12.5*, TAUF =*E12.5
     5    *, TAU =*E12.5*, PSIM =*E12.5)
  292 FORMAT(1H0,7X*NO PLASTIC DEFORMATION OCCURS SINCE PSI NEVER *
     1    *EXCEEDS 1.0*)
  293 FORMAT(1H0,7X*ERROR IN INPUT DATA*)
      END
```

```
      FUNCTION PSIFUN(T)
C
C     SUBROUTINE WHICH PROVIDES RINGLOAD PROGRAM WITH VALUES OF PSI.
C
C     EXPONENTIAL DECAY VERSION.
C     THE PULSE IS A LINEAR RISE FOLLOWED BY AN EXPONENTIAL DECAY.
C
      COMMON/PULSE/TO,TM,TF,TAU,PSIM
      PSIFUN=0.0
      IF(T.LT.TO.OR.T.GT.TF) RETURN
      IF(T.GE.TO.AND.T.LT.TM) 304,305
  304 PSIFUN=PSIM*(T-TO)/(TM-TO)
      RETURN
  305 PSIFUN=PSIM*EXPF((TM-T)/TAU)
      END
```

```
      SUBROUTINE PSICOEF(T,SPSI)
C
C     SUBROUTINE WHICH PROVIDES RINGLOAD PROGRAM WITH THE COEFFICIENTS
C         OF THE POWER SERIES EXPANSION OF PULSE SHAPE PSI.
C
C     EXPONENTIAL DECAY VERSION.
C     THE PULSE IS A LINEAR RISE FOLLOWED BY AN EXPONENTIAL DECAY.
C
C     INPUT
C         T (TIME)
C
C     OUTPUT
C         SPSI(N) (NTH DERIVATIVE OF PSI DIVIDED BY N FACTORIAL AND
C             EVALUATED AT T, FOR N = 1,2,3,4)
C
      DIMENSION SPSI(4)
      COMMON/PULSE/TO,TM,TF,TAU,PSIM
      DO 350 I=1,4
  350 SPSI(I)=0.0
      IF(T.LT.TO.OR.T.GT.TF) RETURN
      IF(T.GE.TO.AND.T.LT.TM) 354,355
  354 SPSI(1)=PSIM/(TM-TO)
      RETURN
  355 E=PSIM*EXPF((TM-T)/TAU)
      SPSI(1)=-E/TAU
      SPSI(2)=E/(2.0*TAU**2)
      SPSI(3)=-E/(6.0*TAU**3)
      SPSI(4)=E/(24.0*TAU**4)
      END
```

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